Suppose $\delta=1$. First note that BL not a NE. Specifically, both players have an incentive to deviate. However, notice that MC and TL are stage game NE’s. Then, if we can construct a SPNE, we’ll need to do it by offering a “bad” stage game NE for defecting and a “good” stage game NE for cooperating in the first period. However, the cost for both players to cooperating is 2 in the first period–player 1 can deviate from B to T and get 3 instead of 1, and player 2 can deviate from L to R and get 4 instead of 2. Then, in order to make them weakly better off cooperating, they’d need to both get at least 2 more in the second stage by cooperating than they would by defecting. However, we can see that the two candidate SGNE payoffs differ only by 1 for player 2. Thus it will be impossible to satisfy player 2’s incentives with BL played in the first period.

2.1

Solution

We’ll assume a monotonic bid function $b(r)$, as well as symmetric bidding functions. Then, for a given valuation $v$, we need to choose a bid $a=b(v)$ to maximize the expected payoff: $F(b^{-1}(a))^2(v-a)$

We use the FOC

$$\frac{\partial F(b^{-1}(a))^2(v-a)}{\partial a} = 0$$
$$\frac{\partial F(b^{-1}(a))^2(v-a)}{\partial a} = 0$$

$$-F(b^{-1}(a))^2 + 2F(b^{-1}(a))f(b^{-1}(a))(v-b(v)) \frac{1}{b'(b^{-1}(a))} = 0$$
\[-F(v)^2 + 2F(v)f(v)(v - b(v)) * \frac{1}{b'(v)} = 0\]

we can rewrite this as

\[-\frac{\partial(F(v)^2b(v))}{\partial v} = 2F(v)f(v)v\]

Integrating both sides from 1 to \(v\), we have

\[-\int_1^v \frac{\partial(F(x)^2b(x))}{\partial x} dx = \int_1^v 2F(x)f(x)x dx\]

\[F(v)^2b(v) - F(1)^2b(1) = \int_1^v 2F(x)f(x)x dx\]

\[F(v)^2b(v) - 0^2b(1) = \int_1^v 2F(x)f(x)x dx\]

\[b(v) = \int_1^v 2F(x)f(x)x dx / F(v)^2\]

Plugging in \(F(v) = (v-1)/2\) and solving, we have

\[b(v) = 1/3 + 2v/3\], which is linear in valuation.

2.2

Solution

The equilibrium in weakly dominant strategies is to always bid your true valuation, as has been shown in the notes. An example of an NE where player 2 always wins is one where player 2 always bids above the maximum possible valuation \(\bar{v}\) and player 1 always bids below the minimum possible valuation, \(\underline{v}\). Then player 1 can only win the auction if she bids above \(\bar{v}\), so her payoff is \(v - \bar{v} - C < 0\) and she has no incentive to deviate from bidding below \(\underline{v}\). Player 2 will always win the auction and get \(v - \underline{v} \geq 0\), regardless of his bid, unless his bid is less than player 1’s, in which case he will get 0. Thus, player 2 has no incentive to deviate, either.

3

Solution
Define 1’s probability of betting given queen as $\sigma_Q$, 1’s probability of betting given king as $\sigma_K$, Player 2’s probability of calling as $\sigma$. Note that player 1 must always bet given king, since the payoff is strictly higher than folding regardless of player 2’s action. $\sigma_K = 1$. Player 1’s expected payoff for betting given queen is $- (1 + x)\sigma + (1 - \sigma) = 1 - 2\sigma - \sigma x$.

Can we have player 1 always folding? In this case, player 2 knows player 1 has a king if she bets, and player 2 will fold. But then player 1 has an incentive to bet given queen, since betting with player 2 folding will yield a payoff of 1, vs a payoff of -1 for player 1 folding. Thus player 1 can’t always fold given a queen.

Can player 1 always bet given a queen? In this case, there is a 50/50 chance that player 1 has a queen contingent on betting. Thus player 2’s expected payoff for calling is $(1 + x - 2)/2 = (x - 1)/2$. If $x > 0$, then $(x - 1)/2 \geq -1$, and player 2 will always call. Then player 1 will strictly prefer to fold given queen. Thus player 1 can’t always bet given queen.

What about mixing. We need to find $(\sigma_Q, \sigma)$ making each player indifferent. Given $\sigma_Q$, the
The probability \( \mu \) of player 1 having a king given bet is 
\[
\frac{\text{Pr}[K|\text{Bet}]}{\text{Pr}[\text{Bet}|K]} = \frac{\text{Pr}[\text{Bet}|K] \cdot \text{Pr}[K]}{\text{Pr}[\text{Bet}|K] + \text{Pr}[\text{Bet}|\text{Queen}] \cdot \text{Pr}[\text{Queen}]} = \frac{1/2}{1/2(1+\sigma_Q)} = \frac{1}{(1+\sigma_Q)}.
\]
For player 1 to be indifferent between betting and folding given queen,
\[
1 - 2\sigma - \sigma x = -1
\]
\[
\sigma(2 + x) = 2
\]
\[
\sigma = \frac{2}{2 + x}
\]

In order for player 2 to be indifferent between calling and folding,
\[
\frac{1}{(1 + \sigma_Q)} \cdot -2 + (1 - \frac{1}{(1 + \sigma_Q)}) \cdot (1 + x) = -1
\]
\[
(1 + x) - (3 + x) \cdot \frac{1}{(1 + \sigma_Q)} = -1
\]
\[
(3 + x) \frac{1}{(1 + \sigma_Q)} = 2 + x
\]
\[
\frac{1}{(1 + \sigma_Q)} \cdot 2 + x
\]
\[
\sigma_Q = \frac{3 + x}{2 + x} - 1
\]

Thus \( \{(\sigma_K = 1, \sigma_Q = \frac{3+x}{2+x} - 1), \sigma = \frac{2}{(2+x)}, \mu = \frac{2+x}{3+x} \} \) is the unique WPNE of this game. The expected payoffs if king is drawn are 
\[
(2, -2) + (1, -1)(1 - \sigma) = (1, -1)(1 + \sigma),
\]
so the payoff is 
\[
1 + \frac{2}{(2+x)}
\]
for player 1 and 
\[
-1 - \frac{2}{(2+x)}
\]
for player 2. If queen is drawn, the payoffs are 
\[
(1, -1)(1 - \sigma_Q) + (1, -1)\sigma_Q(1 - \sigma) + (-1 - x, 1 + x)\sigma_Q\sigma,
\]
so the payoffs are -1 for player 1 and 1 for player 2. Thus player 1’s expected payoff is \( 1/(2+x) \) and player 2’s expected payoff is \( -1/(2+x) \). As \( x \) increases, player 2’s payoff asymptotically approaches player 1’s, but for all finite \( x \) it is strictly less, so no value of \( x \) will satisfy the condition of part 3.