1. 2.4

Solution

Suppose player 1 values the good more than player 2, player 1 bids above player 2’s valuation and player 2 bids below player 1’s valuation and player 1’s bid. Then player 1 gets the object at a price below her valuation. Player 1 has no incentive to change her bid, as she will either get the object at the same price as before or not get the object at all, which would be strictly worse. Player 2 has no incentive to change his bid, as, in order to change the outcome, he would have to outbid player 1, who is already bidding above player 2’s valuation.

2.5

Solution

Part 1 is in the class notes. $u_i(x, v_i) - t(v_i) = x(v_i + \sum_{j \neq i} \hat{v}_j - c)$ if $\sum_{j \neq i} \hat{v}_j > c$ and 0 if $\sum_{j \neq i} \hat{v}_j < c$.

4.3

Solution

The IISDS strategy profile is the same as before. Note that we’re really interested in $q(p(q)-c(q))$, and $p(q)=2-q$, $c(q)=1$ means $p(q)-c(q)=1-q$, the same as $p(q)=1-q$, $c(q)=0$. However, there are many more Nash equilibria (The IISDS equilibrium is, of course, also a Nash Equilibrium). Notice that, if $q \geq 1$, then $p(q)=0$ and profit is $\Pi_i(q_i) = q_i(p(q)) = 0$. Thus, if $q_1 \geq 1$ and $q_2 \geq 1$, then no matter what deviation firm i considers, $q_i + q_{-i} \geq 1$ so profit
remains zero. Thus there is no profitable deviation and this is a Nash Equilibrium. This result is only possible because production is assumed to be costless—if there are any costs whatsoever, producing zero will be a profitable deviation.

5.2

Solution

\[ u_i(g_i, G_{-i}) = (1 - g_i) + \sqrt{g_i + G_{-i}} \]

\[ FOC : \frac{\partial u_i(g_i, G_{-i})}{\partial g_i} = -1 + \frac{1}{2\sqrt{g_i + G_{-i}}} = 0 \]

\[ \frac{1}{2} = \sqrt{g_i + G_{-i}} \]
\[ \frac{1}{4} = g_i + G_{-i} \]
\[ \frac{1}{4} - \sum_{j \neq i} g_j = g_i \]

This is our best response for player i. Then the set of possible Nash Equilibria is \( g^* \) such that \( \sum_i g_i = 1/4 \).

5.3

Solution

\[ u_i(g_i, G_{-i}) = (1 - g_i) + 1/i \sqrt{g_i + G_{-i}} \]

\[ FOC : \frac{\partial u_i(g_i, G_{-i})}{\partial g_i} = -1 + \frac{1}{2i\sqrt{g_i + G_{-i}}} = 0 \]

\[ \frac{1}{2i} = \sqrt{g_i + G_{-i}} \]
\[ \frac{1}{4i^2} = g_i + G_{-i} \]
\[ \frac{1}{4i^2} - \sum_{j \neq i} g_j = g_i \]

then, for an interior solution, it must be that

\[ \frac{1}{4i^2} = \sum_j g_j \]
for all i. Clearly, this cannot be true, so the FOC cannot hold for more than 1 player, so either all players will contribute zero or all but one of the players will contribute zero. The player with the most incentive to contribute is player 1, so let’s assume only player 1 contributes. Then we have $g_1 = 1/4$, $g_i = 0$, $i > 1$. Player 1 satisfies their FOC so they have no incentive to deviate, and for all other players $\frac{\partial u_i(g_i, G_{-i})}{\partial g_i} = -1 + \frac{1}{2i\sqrt{g_i + 1/4}}$ so $\frac{\partial u_i(0, G_{-i})}{\partial g_i} = -1 + \frac{1}{2i\sqrt{1/4}} = -1 + \frac{1}{i} < 0$, so the marginal utility increasing public contributions is negative at zero contributions, so they have no incentive to deviate.

5.3

Solution

$$u_i(g_i, G_{-i}) = \alpha_i \ln(1 - g_i) + (1 - \alpha_i) \ln(g_i + G_{-i})$$

$$\text{FOC}: \frac{\partial u_i(g_i, G_{-i})}{\partial g_i} = -\frac{\alpha_i}{1 - g_i} + \frac{(1 - \alpha_i)}{g_i + G_{-i}} = 0$$

$$\alpha_i(g_i + G_{-i}) = (1 - \alpha_i)(1 - g_i)$$

$$\alpha_i g_i + g_i(1 - \alpha_i) + \alpha_i G_{-i} = (1 - \alpha_i)$$

$$g_i = (1 - \alpha_i) - \alpha_i G_{-i}$$

$$g_i - \alpha_i g_i = (1 - \alpha_i) - \alpha_i G_{-i} - \alpha_i g_i$$

$$g_i(1 - \alpha_i) = (1 - \alpha_i) - \alpha_i G$$

$$g_i = 1 - \frac{\alpha_i G}{1 - \alpha_i}$$

$$g_i = 1 - \frac{\alpha_i}{1 - \alpha_i} \sum_j g_j$$

The set of Nash Equilibria are the strategies satisfying these FOCs. Note that we can solve for $G$ by summing both sides over $i$.

$$\sum_i g_i = n - \sum_i \left( \frac{\alpha_i}{1 - \alpha_i} \sum_j g_j \right)$$

$$G = n - \sum_i \left( \frac{\alpha_i G}{1 - \alpha_i} \right)$$
\[ G = n - G \sum_i \frac{\alpha_i}{1 - \alpha_i} \]
\[ G + G \sum_i \frac{\alpha_i}{1 - \alpha_i} = n \]
\[ G(1 + \sum_i \frac{\alpha_i}{1 - \alpha_i}) = n \]
\[ G = n/(1 + \sum_i \frac{\alpha_i}{1 - \alpha_i}) \]

Plugging this into our FOC, we have

\[ g_i = 1 - \frac{n\alpha_i}{(1 - \alpha_i)(1 + \sum_j \frac{\alpha_j}{1 - \alpha_j})} \]