1. 1

2

nodes (including end nodes): 1+2+4+8+16=32-1=31.

Information sets: 1+1+4+4=10

Individual Strategies: 5 information sets and two actions per information set, so the strategies
are set of functions $\{S : \{1, 2, 3, 4, 5\} \rightarrow \{C, D\}\}$ and there are $2^5 = 32$ strategies for each player.

If player 1 plays C at I11, it doesn’t matter what she plays at I14 or I15, and if player 1 plays D at I11, it doesn’t matter what she plays at I12 or I13, so we’ll write strategies of the form $Ca_{12}a_{13}$ and $Da_{14}a_{15}$. Analogously, player 2’s remaining strategies will be of the form $ca_{22}a_{24}$ and $da_{23}a_{25}$.

<table>
<thead>
<tr>
<th></th>
<th>ccc</th>
<th>ccd</th>
<th>cdc</th>
<th>cdd</th>
<th>dcc</th>
<th>dcd</th>
<th>ddc</th>
<th>ddd</th>
</tr>
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<tbody>
<tr>
<td>CCC</td>
<td>4,4</td>
<td>4,4</td>
<td>2,5</td>
<td>2,5</td>
<td>2,5</td>
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<td>0,6</td>
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<td>2,5</td>
<td>3,3</td>
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<td>1,4</td>
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<td>5,2</td>
<td>3,3</td>
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<td>0,6</td>
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<tr>
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<td>5,2</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>1,4</td>
<td>1,4</td>
</tr>
<tr>
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<td>3,3</td>
<td>5,2</td>
<td>3,3</td>
<td>3,3</td>
<td>1,4</td>
<td>3,3</td>
<td>1,5</td>
</tr>
<tr>
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<td>5,2</td>
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<td>4,1</td>
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<tr>
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<td>4,1</td>
<td>6,0</td>
<td>4,1</td>
<td>3,3</td>
<td>1,4</td>
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<tr>
<td>DDD</td>
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<td>2,2</td>
<td>4,1</td>
<td>2,2</td>
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</tbody>
</table>

The unique NE of the reduced form game is (DDD,ddd).

The set of NE of the original game will be permutations of the reduced form equilibrium with the additional strategies that were dropped. Because those dropped strategies are payoff-irrelevant, all possible strategies that include the reduced form behavior are valid NE’s. Denote player 1’s strategy $a_{11}a_{12}a_{13}a_{14}a_{15}$ and player 2’s strategy $a_{21}a_{22}a_{23}a_{24}a_{25}$. Then the set of NE is $\{(s_1, s_2) : s_1 \in \{DCCDD, DCDDD, DDCDD, DDDDD\}, s_2 \in \{dcdcd, dcddd, dddcd, ddddd\}\}$. The equilibrium outcome is DD in both periods. The unique SPNE is (DDDDD,ddddd).

\textbf{Strictly dominated:} CCC is strictly dominated by CDD and DCC is strictly dominated by DDD for both players (with the lower case analogues for player 2). All 6 other strategies remain under iterated elimination of strictly dominated strategies. \textbf{Weakly Dominated:} All strategies beginning with C are weakly dominated by CDD, and all strategies beginning with D are weakly dominated by DDD. Then at the second stage of elimination, we’re left with the following strategies:
amongst the remaining strategies, DDD and ddd strictly dominate CDD and cdd, so the unique strategy profile surviving IEWDS is (DDD,ddd).

Suppose player i plays C in the last iteration along the path of play. They could do strictly better, regardless of what the other player does, if they played D. Thus DD will be played in the last period along the equilibrium path. Given that DD will be played no matter what at time T, at time T-1 we have the same incentives leading to DD at T-1, and so on, so that DD must be played every period.

\[
s_i = \begin{cases} 
  C & \text{if } h_t = (CC,CC,CC,...,CC) \text{ and } t > 1 \\
  D & \text{else}
\end{cases}
\]

2. 1

\[
\begin{array}{ccc}
  c & d_A & d_B \\
  C & 3,3 & -2,4 & -2,4 \\
  D_A & 4,-2 & 2,2 & -1,-1 \\
  D_B & 4,-2 & -1,-1 & 0,0
\end{array}
\]

The Pure NE of this game are \((D_A,d_A)\) and \((D_B,d_B)\). To find mixed strategy NE’s we need to look for mixed strategies for player 2 that make player 1 indifferent between 2 or more actions, and vice versa. Note first that C is strictly dominated for both players, so we all MSNE will have a probability of playing C of 0. Then we need only consider strategies where \(D_A\) is played with probability \(\sigma\) and \(D_B\) is played with probability \(1 - \sigma\). The payoff for player 2 for \(d_A\) is then \(2\sigma - 1 + \sigma = 3\sigma - 1\) and for \(d_B\) it’s \(-\sigma\). Equating these, we find that player 2 is indifferent when \(3\sigma - 1 = -\sigma\) or \(\sigma = 1/4\). Since the game is symmetric, \(((0,1/4,3/4),(0,1/4,3/4))\) is a MSNE, where the inner vectors are the probabilities of playing \(c,d_A\), and \(d_B\), respectively.

2

Strategy: in period 1, play C, in period 2, play \(D_A\) if \(h = (CC)\), else play \(D_B\).

To prove that this is an NE, we must check for deviations, starting with the last period. In period 2, if CC has been played, the other player will play \(D_A\), so \(d_A\) is the best response.
If anything else has been played, the other player will play \(D_B\), so \(d_B\) is the best response. Given this behavior in period 2, should a player defect in period 1? Defecting nets a player 4 +0, while cooperating yields 3+2. 5 > 4 so cooperation is the best response, and the strategy is an SPNE.

3

Strategy: in period 1, play \((D_A, c)\), in period 2,3,and 4, play \(D_A\) if \(h_1 = (D_Ac)\), else play \(D_B\).

To prove that this is an NE, we must check for deviations. Note that, after the first period, all strategies are NE of the stage game, so the subgame perfect requirement is satisfied. Given this behavior in later periods, should either player defect in period 1? Note that the players’ strategies and equilibrium outcomes are symmetric except for the first period, where player 2 is worse off. Then if player 2 has no incentive to deviate, neither will player 1, so we may consider only player 2’s problem. Defecting to \(d_A\), the best response of the stage game given \(D_A\), nets player 2+0+0+0=2, while cooperating yields -2+2+2+2=4. 4 > 2 so cooperation is the best response, and the strategy is an SPNE.

4

Strategy: in period 1, play \((D_A, d_B)\). In period 2, if \(h_1 = (D_Ad_B)\), play \((D_B, d_A)\). In later periods, play \(D_A\) if \(h_2 = (D_Ad_B, D_Bd_A)\), else play \(D_B\).

To prove that this is an NE, we must check for deviations. Note that, after the second period, all strategies are NE of the stage game, so the subgame perfect requirement is satisfied for all histories after period 2. Given this behavior in later periods, should either player defect in period 2? Recall that we can consider only 1-shot deviations, so considering 1-shot deviations in period 1 and 2 will be sufficient to show SPNE. Consider player 2’s problem. In period 2, defecting to \(d_B\), the best response of the stage game given \(D_B\), nets player -1+0+0+0+0+...+0=-1, while cooperating yields -1-1+2+2+...+2=2(T-2)-2=2T-6. If \(T > 2\), cooperating is a best response. For player 1 in period 2, defecting to \(D_A\), the best response of the stage game given \(D_B\), nets player 2+0+0+0+0+...+0=2, while cooperating yields -1+1+2+2+...+2=2(T-2)-2=2T-6. If \(T > 2\), cooperating is a best response. Now let’s look at deviations in period 1. In period 1, defecting to \(d_A\), the best response of the stage game given \(D_A\), nets player 2+0+0+0+0+...+0=2, while cooperating yields -1+1+2+2+...+2=2(T-2)-2=2T-6. If \(T > 3\), cooperating is a best response. For player 1 in period 1, defecting to \(D_B\), the best response of the stage game given \(d_B\), nets player 0+0+0+0+0+...+0=0, while cooperating yields -1+1+2+2+...+2=2(T-2)-2=2T-6. If \(T > 2\), cooperating is a best response. Thus if \(T > 3\), cooperating is always (weakly) a best response.
3. 1

By our second Folk Theorem proposition, \((CC, CC, \ldots)\) is supportable as an SPNE outcome if

\[
\frac{2}{(1 - \delta)} \geq 3 + \frac{\delta}{(1 - \delta)}
\]

\[
2 \geq 3(1 - \delta) + \delta
\]

\[
2 \geq 3 - 3\delta + \delta
\]

\[
-1 \geq -2\delta
\]

\[
1/2 \leq \delta
\]

The strategy itself is

\[
s_i = \begin{cases} 
  C & \text{if } h_t = (CC, CC, CC, \ldots, CC) \\
  D & \text{else}
\end{cases}
\]

The unique stage game NE is \(Dd\), so by our second Folk Theorem proposition, \((Cc, Cc, \ldots)\) is supportable as an SPNE outcome if

\[
\frac{2}{(1 - \delta)} \geq 3 + \frac{\delta}{(1 - \delta)}
\]

\[
2 \geq 3(1 - \delta) + \delta
\]

\[
2 \geq 3 - 3\delta + \delta
\]

\[
-1 \geq -2\delta
\]

\[
1/2 \leq \delta
\]

The strategy itself is

\[
s_i = \begin{cases} 
  C & \text{if } h_t = (CC, CC, CC, \ldots, CC) \\
  D & \text{else}
\end{cases}
\]

Note this is exactly the same as before. All we’ve done is add the \(P\) strategy, which is not a part of any stage game NE and is strictly dominated by \(D\), precluding its use in a profitable deviation. Thus \(P\) doesn’t change our options for reversion strategies.
In the first game, D is the dominant strategy, and the worst payoff you can get from that strategy is 1. Thus the minmax values for the first game are (1,1). In the second game, the best you can do if your opponent plays P is to play D and get zero. If your opponent plays C or D, you can get positive payoffs. Thus the worst payoff your opponent can force on you is 0, and the minmax values are (0,0).

\[
\text{Strategy: play } s_1 = \begin{cases} 
C & \text{if } h_t = (CC, CC, CC, \ldots, CC) \\
P & (h_{t-2} = (CC, CC, \ldots, CC) \land (a_{t-1} = Dc)) \\
D & (h_{t-2} = (CC, CC, \ldots, CC) \land (a_{t-1} = Cd)) \\
\text{else} & D
\end{cases}
\]
\[
\text{Strategy: play } s_2 = \begin{cases} 
C & \text{if } h_t = (CC, CC, CC, \ldots, CC) \\
P & (a_{t-1} = Dc) \\
D & (a_{t-1} = Cd) \\
\text{else} & \begin{cases} 
P & \text{if } P \text{ or } P \in h_t \\
D & \text{if } P \text{ or } P \in h_t
\end{cases}
\end{cases}
\]

To prove that this is an NE, we must check for deviations. Let’s start with player 1. Player 1 can either be 1) playing on the equilibrium path, 2) playing after a first deviation to Dc, 3) playing after a first deviation to Cd, 4) playing after Pd or Dp has been played or p has been played in the two last period, 5) playing after p has been played in the the last period but not the period before, 6) playing when Dd is the equilibrium path, and 7) playing at other histories. We’ll check for 1 shot deviations in each of these cases. In 1), the payoff for cooperating (ignoring previous periods, as they won’t be affected by a deviation either way) is \(2/(1-\delta)\), and the payoff for the best deviation of D is \(3-\delta+\delta^2+\delta^3+\ldots = 2-2\delta+1/(1-\delta)\). This is supportable if

\[
\frac{2}{1-\delta} \geq 2 - 2\delta + \frac{1}{1-\delta}
\]

which holds when \(\delta \geq 1 - \sqrt{2}/2\).

In 2), the payoff for playing the strategy (ignoring previous periods, as they won’t be affected by a deviation either way) is \(-2 + 1/(1-\delta)\), and the payoff for the best deviation of D is
\[1 - \delta - \delta^2 + \delta^3 + \delta^4 + ... = -2\delta - 2\delta^2 + 1/(1 - \delta).\] This is supportable if
\[-2 + 1/(1 - \delta) \geq -2\delta - 2\delta^2 + 1/(1 - \delta)\]
\[1 \leq \delta + \delta^2\]
which holds when \(\delta \geq 1/2(\sqrt{5} - 1).\)

In 3), there is no profitable deviation because D is the best response of the stage game.

In 4), the payoff for playing the strategy (ignoring previous periods, as they won’t be affected by a deviation either way) is \(-2 + 1/(1 - \delta),\) and the payoff for the best deviation of D is \(1 - \delta - \delta^2 + \delta^3 + \delta^4 + ... = -2\delta - 2\delta^2 + 1/(1 - \delta).\) This is supportable if
\[-2 + 1/(1 - \delta) \geq -2\delta - 2\delta^2 + 1/(1 - \delta)\]
\[1 \leq \delta + \delta^2\]
which holds when \(\delta \geq 1/2(\sqrt{5} - 1).\)

In 5), the payoff for playing the strategy (ignoring previous periods, as they won’t be affected by a deviation either way) is \(-2 + 1/(1 - \delta),\) and the payoff for the best deviation of D is \(-\delta - \delta^2 + \delta^3 + \delta^4 + ... = -1 - 2\delta - 2\delta^2 + 1/(1 - \delta).\) This is supportable if
\[-2 + 1/(1 - \delta) \geq -1 - 2\delta - 2\delta^2 + 1/(1 - \delta)\]
\[1/2 \leq \delta + \delta^2\]
which holds when the condition in 4) holds.

In 6), the best response of the stage game is to play D, so there is no incentive to deviate.

In 7), playing the strategy will require playing P twice, so the payoff is \(-2 - 2\delta + 1/(1 - \delta),\) and the payoff for the best deviation of D is \(0 - \delta - \delta^2 + \delta^3 + \delta^4 + ... = -1 - 2\delta - 2\delta^2 + 1/(1 - \delta).\) This is supportable if
\[-2 - 2\delta + 1/(1 - \delta) \geq -1 - 2\delta - 2\delta^2 + 1/(1 - \delta)\]
\[1 \leq 2\delta^2\]
which holds when \(\delta \geq \sqrt{2}/2.\) This is the strictest requirement on \(\delta,\) and thus and \(\delta\) satisfying the criterion can support this strategy profile as a SPNE.