1.1 For periods $1 + 15 \cdot k$ to $2 + 15 \cdot k$, $k \in \mathbb{N}$, Player 1 plays D, player 2 plays c if history is Dc for all periods $1 + 15 \cdot k$ to $2 + 15 \cdot k$, $k \in \mathbb{N}$, Dd for periods $3 + 15 \cdot k$, $k \in \mathbb{N}$ and Cc for all periods $4 + 15 \cdot k$ to $15 + 15 \cdot k$; else both players play d. For periods $3 + 15 \cdot k$, $k \in \mathbb{N}$, Player 1 plays D, player 2 plays d if history is Dc for all periods $1 + 15 \cdot k$ to $2 + 15 \cdot k$, $k \in \mathbb{N}$, Dd for periods $3 + 15 \cdot k$, $k \in \mathbb{N}$ and Cc for all periods $4 + 15 \cdot k$ to $15 + 15 \cdot k$, else both players play d.

For periods $4 + 15 \cdot k$ to $15 + 15 \cdot k$, $k \in \mathbb{N}$, Player 1 plays C and player 2 plays c if history is Dc for all periods $1 + 15 \cdot k$ to $2 + 15 \cdot k$, $k \in \mathbb{N}$, Dd for periods $3 + 15 \cdot k$, $k \in \mathbb{N}$ and Cc for all periods $4 + 15 \cdot k$ to $15 + 15 \cdot k$, else both players play d.

Player 2 is the only player who plays an action where she has a profitable deviation, so let’s consider her problem. Player 2’s payoff for deviating is always the same on the equilibrium path, but the payoff for sticking to the strategy is lowest at the first period where she plays c while player 1 plays D. The payoff here is $\sum_{t=2}^{2} \delta^t 1 + \sum_{t=3}^{14} \delta^t 2 + \sum_{t=17}^{17} \delta^t 1 + \sum_{t=18}^{29} \delta^t 2 + \ldots$ This is less than $\sum_{t=2}^{2} \delta^t 1 + \sum_{t=3}^{14} \delta^t 2 + \sum_{t=17}^{17} \delta^t 1 + \sum_{t=18}^{29} \delta^t 2 + \ldots$ Define $\delta^* = \delta^{15}$. Then we have $\sum_{t=1}^{\infty} 1 \delta^{st} + \sum_{t=1}^{\infty} 24 \delta^{st} = 25 \delta^{15}/(1 - \delta^{15})$. Deviating to c, we have Dd forever, a payoff of $\sum_{t=0}^{\infty} \delta^t = 1/(1 - \delta)$. For $\delta$ sufficiently large, the continuation payoff is approximately $25/(1 - \delta^{50})$, which is greater than $25/15/(1 - \delta)$, which is greater than $1/(1 - \delta)$, so there is no profitable deviation.

1.2

For periods $1 + 50 \cdot k$ to $21 + 50 \cdot k$, $k \in \mathbb{N}$, Player 1 plays D, player 2 plays c if history is Dc for all periods $1 + 50 \cdot k$ to $21 + 50 \cdot k$ and Cd for all periods $22 + 50 \cdot k$ to $50 + 50 \cdot k$, else player 2 plays d.

For periods $22 + 50 \cdot k$ to $50 + 50 \cdot k$, $k \in \mathbb{N}$, Player 1 plays C, player 2 plays d if history is Dc for all periods $1 + 50 \cdot k$ to $21 + 50 \cdot k$ and Cd for all periods $22 + 50 \cdot k$ to $50 + 50 \cdot k$, else player 1 plays D.
The best time to deviate will be at the beginning of a run of cooperate, when you have the most to lose by sticking to the strategy. Let’s consider player 1, who has to cooperate more often and thus has a greater incentive to deviate. Given $\delta$, the payoff for continuing is at the beginning of player 1’s cooperation is $\sum_{t=29}^{50} \delta^5 t + \sum_{t=79}^{100} \delta^5 t + ...$. This is less than $\sum_{t=29}^{50} \delta^{50} 5 + \sum_{t=79}^{100} \delta^{100} 5 + ...$. Define $\delta^* = \delta^{50}$. Then we have $\sum_{t=1}^{\infty} 105\delta^* t = 105\delta^{50}/(1-\delta^{50})$. Deviating to D, we have Dd forever, a payoff of $\sum_{t=0}^{\infty} \delta^t = 1/(1-\delta)$. For $\delta$ sufficiently large, the continuation payoff is approximately $105/(1-\delta^{50})$, which is greater than $105/50/(1-\delta)$, which is greater than $1/(1-\delta)$, so there is no profitable deviation.

1.3

Consider 3 phases:

1) Cooperative face. In this phase play cc

2) Punish 1/reward 2. In this phase play cd for 2 periods before returning to the cooperative phase

3) Punish 2/reward 1. In this phase play dc for 2 periods before returning to the cooperative phase

A) If in the cooperative phase and cc is played, stay in the cooperative phase.

B) If in punish 1 and 1 deviates. Start over the punish 1 phase.

C) If in punish 1 and 2 deviates. Ignore it.

D) If in punish 2 and 2 deviates. Start over the punish 2 phase.

E) If in punish 2 and 1 deviates. Ignore it.

F) If in any punishment phase and it is not started over, go to the next period in the punishment phase if it has been played for less than 2 periods. If it is the final period of the punishment phase, return to the cooperative phase.

Subgame perfection: On the equilibrium path, the payoff is $2/(1-\delta)$. Deviating yields $2/(1-\delta) - 2 - 2\delta - 2\delta^2 + 5 + 0 + 0$. Then to be subgame perfect, we need

$$2/(1-\delta) \geq 2/(1-\delta) - 2 - 2\delta - 2\delta^2 + 5$$

$$2\delta + 2\delta^2 \geq 3$$

$$\delta \geq (\sqrt{7} - 1)/2 \approx .82$$

Off the equilibrium path, the punisher has no incentive to deviate, since they’re action doesn’t affect the evolution of the game and D has a higher stage game payoff. The agent being punished can cooperate and, in the worst case scenario, will have to accept another period of punishment
and then go back to CC, a payoff of \(\frac{2}{(1 - \delta)} - 2 - 2\delta\). Deviating to D at this history will result in a payoff of \(\frac{2}{(1 - \delta)} - 2\delta - 2\delta^2 + 1 - 2\). Then to be subgame perfect, we need

\[
\frac{2}{(1 - \delta)} - 2 - 2\delta \geq \frac{2}{(1 - \delta)} - 2\delta - 2\delta^2 + 1 - 2
\]

\[
2\delta^2 \geq 1
\]

\[
\delta \geq 1/\sqrt{2} \approx 0.71
\]

Thus we have subgame perfection. What about internal consistency? In the punishment phase, the punisher is strictly better off than in the cooperate phase (5 in the next 1 or 2 periods, then back to 2, vs 2 in all future periods) or the phase where she would be punished (5 in next 1 or 2 periods vs 0 in next 1 or 2 periods, and 2 forever after). Being in the first stage of the punishment phase rather than the second is also better for the punisher and worse for the one being punished. In particular, the set of all possible history dependent payoffs for the next 2 periods (after that, you go back to CC forever) is, during cooperation \((2 + 2\delta, 2 + 2\delta)\), 1st period punishment of player 1 \((0, 5 + 5\delta)\), 2nd period punishment of player 1 \((2\delta, 5 + 2\delta)\), and symmetric payoffs for punishment of player 2. None of these Pareto dominates any other, so internal consistency is satisfied.

\[ \text{3.2} \]

The solution provided in the other notes assumes a pure strategy equilibrium. Mixed strategies cannot be part of the equilibrium path play, since D strictly dominates C for both players. Thus on the path of play it is never optimal to play a mixed strategy, no matter what your opponent does. However, for the three information sets each player has, the set of Nash equilibria is of the form \(Da_2a_3\), where the actions in the two later periods can be arbitrary. Thus, these off path actions can include any mixed strategy. Then given \(\sigma_i\) being the probability of playing C at information set i, the set of MSNE are \{\((D\sigma_{i1}\sigma_{i2}, D\sigma_{i1}\sigma_{i2}) : \sigma_{ij} \in [0, 1]\}\}.

\[ \text{3.3} \]

As with 3.2, we get the same MSNE, except that now actions in T-1 later information sets can be any mixed strategy between C and D.