Econ 711 Homework 8 Solutions

April 25, 2014

1.1-1.8
See class notes

1.9

Solution

Note that \( f \) is the value of \( x_B \) necessary to make \((x_G, x_B) \sim x^* \sim m. \) \( h \) is the full insurance payoff such that \((h(x_G), h(x_G)) \sim_h (x_G, f(x_G)). \) Since \( x_G \in [x^*, m_G], u(x_G) < u(m_G) \) and \( u(f(x_G)) > u(m_B). \) Since

\[
\pi_L u(f(x_G)) + (1 - \pi_L)u(x_G) = \pi_L u(m_B) + (1 - \pi_L)u(m_G)
\]

we have

\[
\pi_L (u(f(x_G) - u(m_B)) = (1 - \pi_L)(u(m_G) - u(x_G))
\]

Since \( \pi_H > \pi_L, \) it must be that

\[
\pi_H (u(f(x_G) - u(m_B)) > (1 - \pi_H)(u(m_G) - u(x_G))
\]

\[
u(h(x_G)) = \pi_H u(f(x_G)) + (1 - \pi_H)u(x_G) > \pi_H u(m_B) + (1 - \pi_H)u(m_G)\]

. Thus \( IR_H \) is satisfied. \( IR_L \) binds by the definition of \( f. \) \( IC_H \) binds by the definition of \( h. \) Finally, \( IC_L \) is satisfied since the utility of full insurance is the same for both high and low types, so \( u(h(x_G)) = \pi_H u(f(x_G)) + (1 - \pi_H)u(x_G) < \pi_L u(f(x_G)) + (1 - \pi_L)u(x_G) \) since the payoff in the good state is higher than the payoff in the bad state. Thus this pair of contracts statisfies all our
Incentive and rationality constraints. Is it more profitable? The high type customer gets strictly less in both periods, than with contract \( x^* \), since

\[
u(h(x_G)) = \pi_H u(f(x_G)) + (1 - \pi_H) u(x_G) < \pi_L u(f(x_G)) + (1 - \pi_L) u(x_G) = u(x^*)\]

Thus the firm gets more profit. The low type customer gets \( u(x^*) = \pi_L u(f(x_G)) + (1 - \pi_L) u(x_G) \leq u(\pi_L f(x_G) + (1 - \pi_L) x_G) \). Thus it must be that

\[
x^* \leq \pi_L f(x_G) + (1 - \pi_L) x_G
\]

Firm Profit: \( \alpha(\pi_L(m_B - f(x_G)) + (1 - \pi_L)(m_G - x_G)) + (1 - \alpha)(\pi_H(m_B - h(x_G)) + (1 - \pi_H)(m_G - h(x_G))) \)

\[
\frac{\partial}{\partial x_G} \alpha(\pi_L(m_B - f(x_G)) + (1 - \pi_L)(m_G - x_G)) + (1 - \alpha)(\pi_H(m_B - h(x_G)) + (1 - \pi_H)(m_G - h(x_G)))
\]

\[
= \alpha(\pi_L - \pi_L f'(x_G) - 1) - (1 - \alpha)h'(x_G)
\]

Evaluating this at \( x_G = x^* \), note that \( f(x^*) = x^* \). \( h'(x^*) \) is strictly negative (good period payoffs matter relatively less for the high type player, who gets good draws less frequently, so increasing good payouts and decreasing bad payouts such that the low type will be indifferent with \( x^* \) will make the high type worse off. Thus the high type can be offered an \( h(x_G) < x^* \) and \( h \) is decreasing). We want to evaluate the sign of

\[
\alpha(\pi_L - \pi_L f'(x^*) - 1) - (1 - \alpha)h'(x^*)
\]

What is \( f'(x^*) \)? Let’s go back to the equation that defines it. \( \pi_L u(f(x_G)) + (1 - \pi_L) u(x_G) = u(x^*) \). Solving for \( f \) will be a pain, given the undefined \( u \) function. However, Evaluating at \( x_G = x^* \) lets us use the a first order taylor series approximation. Define \( f(x^* + \epsilon) = x^* - \delta(\epsilon). \) Then \( u(x^* + \epsilon) \approx u(x^*) + u'(x^*) \epsilon \) and \( u(x^* - \delta(\epsilon)) \approx u(x^*) - u'(x^*) \delta(\epsilon). \) Thus we have

\[
\pi_L(u(x^*) - u'(x^*) \delta(\epsilon)) + (1 - \pi_L)(u(x^*) + u'(x^*) \epsilon) \approx u(x^*)
\]

\[
\pi_L(-u'(x^*) \delta(\epsilon)) + (1 - \pi_L)(u'(x^*) \epsilon) \approx 0
\]

\[
(1 - \pi_L)(u'(x^*) \epsilon \approx \pi_L(u'(x^*) \delta(\epsilon))
\]

\[
\frac{(1 - \pi_L)}{\pi_L} \epsilon \approx \delta(\epsilon)
\]

2
Thus \( f'(x^*) = -\frac{(1-\pi L)}{\pi L} \), so we have

\[
\begin{align*}
&= \alpha(\pi L - \pi_L f'(x^*) - 1) - (1-\alpha)h'(x^*) = -(1-\alpha)h'(x^*) > 0
\end{align*}
\]

2.1

Solution

This is given by theorem 2 in the notes. Also, we see that, if the NE is in pure strategies and \( g \) maps onto \( A \) (that is, \( g \) maps only to elements of \( \Delta A \) with all probability on a single outcome), \( \phi(a, \theta) = g(a, s^* (\theta)) = 1 \) for some \( a \) and 0 for all other \( a \)’s. That is, \( \phi : \theta \to A \), so the truthtelling mechanism is also deterministic. Note that I’ve changed \( \sigma \) in the notes to \( \phi \) to avoid notational issues in the next problem.

2.2

Solution

Suppose \( \sigma^* \) is a BNE of the game \( \langle M, g \rangle \). Let \( \phi(a, \theta) = \sum_{m \in M} g(a, m_i, m_{-i}) \sigma^* (m_i | \theta_i) \prod_{j \neq i} \sigma^* (m_j | \theta_j) \).

For any \( \hat{\theta}_i \in \Theta_i \), take \( \hat{\sigma}_i = \sigma^*_i (\hat{\theta}_i) \). We can write agent \( i \)’s interim payoffs as

\[
\begin{align*}
v_i(\hat{\sigma}_i, \sigma_{-i}^*, \theta_i) &= \sum_{\theta_{-i}} \sum_{m_i \in M_i} \sum_{m_{-i} \in M_{-i}} \sum_{a} u_i(a, \theta) g(a, m_i, m_{-i}) Pr(\theta_{-i} | \theta_i) \sigma^*_i (m_i | \hat{\theta}_i) \prod_{j \neq i} \sigma^*_j (m_j | \theta_j) \\
&= \sum_{\theta_{-i}} \sum_{a} u_i^{DR}(a, \hat{\theta}_i, \theta_{-i}, \theta) Pr(\theta_{-i} | \theta_i) \\
&= \sum_{\theta_{-i}} u_i^{DR}(a, \hat{\theta}_i, \theta_{-i}, \theta) Pr(\theta_{-i} | \theta_i).
\end{align*}
\]

Since \( \sigma^* \) is a NE of the original game, \( v_i(\sigma^*_i, \sigma_{-i}^*, \theta_i) \geq v_i(\hat{\sigma}_i, \sigma_{-i}^*, \theta_i) \) for all \( i, \theta_{-i}, \) and \( \sigma_{-i}^*(\hat{\theta}_i) \) at \( \hat{\theta}_i \in \Theta_i \). Thus,

\[
\sum_{\theta_{-i}} u_i^{DR}(a, \hat{\theta}_i, \theta_{-i}, \theta) Pr(\theta_{-i} | \theta_i) = v_i(\sigma^*_i, \sigma_{-i}^*, \theta_i) \geq v_i(\hat{\sigma}_i, \sigma_{-i}^*, \theta_i) = \sum_{\theta_{-i}} u_i^{DR}(a, \hat{\theta}_i, \theta_{-i}, \theta) Pr(\theta_{-i} | \theta_i)
\]

Hence truthtelling is Nash in the direct mechanism.
Solution

This is a corollary to problem 2. Simply note that a direct mechanism is a type of mechanism and non-truth telling is a type of strategy.

4.1

Solution

Not generally. A given player may have two or more potential reports that yield the same outcome. For example, if there are two policies and increasing type yields increasing utility in one policy and no change in the utility from the other (e.g. the policy of doing nothing), then if player i wants the active policy and doesn’t get it by reporting their true type, they could report a lower type and get the same payoff, since the policy still wouldn’t be implemented. Conversely, if they want the policy to be implemented and it is, reporting a higher type will not change the outcome, and thus won’t change their payoff.

4.2

Solution

Yes. Given reports $\hat{\theta}_{-i}, \hat{\theta}_i = \theta_i$ is the weakly dominate strategy. to see this, consider the payoff for agent i.

$$\pi_i(\hat{\theta}_i, \hat{\theta}_{-i}) = u_i(d(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) - (C(d(\hat{\theta}_i, \hat{\theta}_{-i})) - \sum_{j \neq i} u_j(d(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_j))$$

$$= u_i(d(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) + \sum_{j \neq i} u_j(d(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_j)) - C(d(\hat{\theta}_i, \hat{\theta}_{-i}))$$

and note that d is such that $d(x) \in \arg\max_{x \in X} \sum_{i=1}^n u_i(x, \hat{\theta}_i) - C(x)$.

Thus individual utility depends on own report only to the extent that one’s own report may change the optimal x that’s chosen, and, given a report of $\theta_i$, d will maximize $u_j(x, \theta_i) + \sum_{j \neq i} u_j(x, \hat{\theta}_j) - C(x)$, which is in fact agent i’s payoff function.

4.3
Solution

\[ d(\theta) \in \arg\max_{p \in X} \sum_{i=1}^{n} u_i(p, \theta_i) \]

\[ t_i(\theta) = -\sum_{j \neq i}^{n} u_j(d(\theta), \theta_j) \]

\[ \pi_i(\theta_i, \theta_{-i}) = \sum_{i=1}^{n} u_i(p, \theta_i) \]. Suppose \( u_i(p, \theta_i) = p_i u_i(\theta_i) \). Then this mechanism simply chooses the optimal probability distribution for the (costless) allocation of an indivisible good, in terms of aggregate utility. The maximal p will be to award the good to the highest utility agent with probability 1.

4.4

Solution

\[ d(\theta) \in \arg\max_{p \in [0,1]} \sum_{i=1}^{n} u_i(p, \theta_i) \]

\[ t_i(\theta) = cp - \sum_{j \neq i}^{n} u_j(d(\theta), \theta_j) \]

\[ \pi_i(\theta_i, \theta_{-i}) = \sum_{i=1}^{n} u_i(p, \theta_i) - cp \]. Suppose \( u_i(p, \theta_i) = pu_i(\theta_i) \). Then this mechanism can be interpreted as the binary policy groves mechanisms where the social planner can choose any probability of provision, rather than just 0 or 1 (this will never really matter, though. Proper mixing will only occur when the cost of the policy equals the utility it gives). Alternatively, it could be interpreted as the intensity of the policy.