Consider the model of monopoly provision of insurance under adverse selection considered in class. Let expected utility be \( J u(x_B) + (1 - J) u(x_G) \) where \((x_G, x_B)\) are the consumptions in the good and bad states respectively and \(J\) is the probability of the bad state for types \(L, H\). Assume that \(J_L < J_H\) and let the endowment be \((m_G, m_B)\) with \(m_B < m_G\).

1. Assuming that the insurance provider is risk neutral set up and solve the relevant problem in case the type is observable. Explain why this solution is not implementable when the insurance provider cannot observe the type.

2. Set up the problem for the monopolist who cannot observe type.

3. Prove that \(x^L_G \geq x^H_G\).

4. Prove that \(x^L_B \leq m_G\).

5. Prove that the incentive compatibility constraint for type \(H\) must bind.

6. Prove that the individual rationality constraint for type \(L\) must bind.

7. Prove that \(x^L_G > x^L_B\).

8. Prove that \(x^H_G = x^H_B\).

9. Prove that \(x^L_G > x^H_G\). To do this, suppose not in which case the solution is to pool the two types on \((x_G, x_B)\) = \(x^*\) where

\[
u(x^*) = \pi_L u(m_B) + (1 - \pi L) u(m_G).
\]

Let \(f(x_G)\) be such that

\[
\pi_L u(f(x_G)) + (1 - \pi L) u(x_G) = u(x^*) = \pi_L u(m_B) + (1 - \pi L) u(m_G)
\]

and \(h(x_G)\) solve

\[
u(h(x_G)) = \pi_L u(f(x_G)) + (1 - \pi L) u(x_G).
\]

Argue that, for \(x_G \in [x^*, m_G]\) type \(H\) would pick \((x^H_G, x^H_B)\) = \((h(x_G), h(x_G))\) and type \(L\) would pick \((x^L_G, x^L_B)\) = \((x_G, f(x_G))\) if these are the two contracts offered. Take the derivative with respect to \(x_G\) and show that the profit increases when evaluating at \(x_G = x^*\).

2. Let \((M, g)\) be a mechanism and \(u_i(a, \theta)\) be the utility function for \(i = 1, \ldots, n\).

1. Suppose the mechanism is deterministic so that \(g : M \rightarrow A\) and let \(A\) be finite. Prove that if \(s^*\) is a pure strategy Nash equilibrium in the Bayesian game induced by \((M, g)\) there is direct mechanism that generates the same allocation such that truth-telling is a Bayesian Nash equilibrium in the game induced by the direct mechanism.

2. Adjust the proof above to show that if \(\sigma^*\) is a mixed strategy Nash equilibrium in the Bayesian game induced by \((M, g)\) there is direct mechanism that generates the same allocation such that truth-telling is a Bayesian Nash equilibrium in the game induced by the direct mechanism.

3. Prove that if a direct mechanism \((\Theta, g)\) has an equilibrium in which some agents misrepresent their preferences, then there exists an alternative direct mechanism \((\Theta, \tilde{g})\) that generates the same allocation as a truth-telling equilibrium.
4. Suppose that \( N = \{1, \ldots, n\} \), and that the set of allocations is \( A = X \times R^n \) where \( x \in X \) can be interpreted as some social decision and \( t \in R^n \) is a vector of transfers. Let \( \theta_i \in \Theta_i \) be the type of agent \( i \) and assume that preferences are \( u_i (\theta_i, x) - t_i \). Assume that \( x \) costs \( C(x) \) “dollars” to implement. Let a direct revelation mechanism \((d, t) : \times_{i=1}^n \Theta_i \rightarrow X \times R^n \) be given by

\[
d(\theta) \in \arg\max_{x \in X} \sum_{i=1}^n u_i (x, \theta_i) - C(x)
\]

\[
t_i (\theta) = C(d(\theta)) - \sum_{j \neq i} u_i (d(\theta), \theta_j)
\]

1. Is trut-telling a strictly dominant strategy?
2. Is truth-telling a weakly dominant strategy?
3. For the case with \( X = \{ p \in R^n_+ | \sum_{i=1}^n p_i \leq 1 \} \) and \( C(p) = 0 \) for every \( p \) what does the mechanism simplify to?
4. For the case with \( X = \{ p \in [0, 1] \} \) and \( C(p) = pc \) for every \( p \) what does the mechanism simplify to?