1 Extensive Form Games

Definition 1 A finite extensive form game is an object $K = \{N, (T \prec), P, A, H, u, \rho\}$ where:

- $N = \{0, 1, ..., n\}$ is the set of agents (player 0 is “nature”)
- $(T \prec)$ is the game tree
- $P$ is the player partitioning
- $A$ is the set of actions
- $H$ is the informational partitioning
- $u : Z \to R^n$ is the payoff function, where $Z \subseteq T$ denotes the set of endnodes.
- $\rho$ is the probability distribution over moves by nature.

We will now fill in the details:

Definition 2 A finite game tree $(T, \prec)$ is a finite set of nodes $T$ and a binary relation $\prec$ on $T$ (“precedes”) with the following properties:

1. $\prec$ is asymmetric. There exists no pair $(t', t'')$ such that $t' \prec t''$ and $t'' \prec t'$. 
2. $\prec$ is transitive. If $(t', t'', t''')$ satisfy $t' \prec t''$ and $t'' \prec t'''$, then $t' \prec t'''$.
3. Each node is a complete description of the history. For any $(t', t'', t''')$, if $t' \prec t'''$ and $t'' \prec t'''$ then either $t' \prec t''$ or $t'' \prec t'$.

To understand the need for the third property suppose that neither $t' \prec t''$ nor $t'' \prec t'$ despite both nodes preceding $t'''$ (draw!). Then, there are two distinct sequences of nodes that both “merge” at some point before reaching $t'''$, which is something we don’t allow for a game tree.

A more direct definition would be:
Definition 3 Given a pair $(T, \prec)$ where $\prec$ is asymmetric and transitive we say that each node is a complete description of the history if for any $t$ such that the set of preceding nodes is non-empty there exists a unique sequence $\{t_k\}_{k=1}^K$ with $t_k \in T$ for each $k$ such that $t_1 \prec t_2 \prec \ldots t_K \prec t$.

You will show that these properties are equivalent as an exercise.

Definition 4 The set of terminal nodes of $(T, \prec)$ is given by $Z = \{t \in T | \exists t' \in T \text{ such that } t \prec t'\}$.

Definition 5 $P = \{P_i\}_{i=1}^n$ is a player partitioning if it is a partition of $T \setminus Z$.

A more direct way to assign “ownership” to nodes is to map the set of non-terminal nodes onto the set of players, so that

$$i : T \setminus Z \rightarrow \{0, 1, \ldots, n\}.$$ 

Then the player partition can be generated by the inverse images to $i$ so that

$$P_j = i^{-1}(j)$$

for each $j$.

Definition 6 For every $t \in T \setminus Z$ let

$$S(t) = \{t' \in T | t \prec t' \text{ and } \exists t'' \in T \text{ such that } t \prec t'' \prec t'\}$$

be the immediate successors to $t$.

Definition 7 For each $t \in T \setminus Z$ let $A(t)$ be a finite set of actions available at node $t$ and let $\alpha(t) : A(t) \rightarrow S(t)$ be a one to one function that describes how actions are mapped to successor nodes. Let $A = \{A(t)\}_{t \in T \setminus Z}$ denote the set of all actions available in the game.

Definition 8 The information partition $H = \{H_j\}_{j=1}^n$ is a partition of $T \setminus Z$ such that each $H_j$ is a partition of $P_j$ satisfying:
1. For every $h_j \in H_j$ and $(t', t'') \in h_j$ neither $t' < t''$ nor $t'' < t'$

2. For every $h_j \in H_j$ and $(t', t'') \in h_j$ we have that $A(t') = A(t'') \equiv A(h_j)$

It is immediate that information sets don’t overlap, as each $H_j$ is a partition.

**Definition 9** An extensive form payoff function is an object $u = (u_1, ..., u_n)$ where $u_i : Z \rightarrow R$ for each $i$.

We will assume that nature only moves at the root of the game, which is without loss of generality.

**Definition 10** The probability distribution over moves by nature is some $\rho \in \Delta(S(t_0))$, where $t_0$ is the unique node so that $t_0 < t$ for any $t \in T \setminus \{t_0\}$ and $S(t_0)$ are the immediate successors to $t_0$.

### 1.1 Perfect Recall

The general definition of an extensive form game allows for players that forget what they did in the past. This is usually ruled out:

**Definition 11** $K$ is said to be an extensive form game of perfect recall if for every $h_j' \in H_j$, every $(t', t'') \in h_j'$, and every $t \in P_j$ such that $t < t'$ there exists some node $\hat{t}$ (possibly $\hat{t} = t$) and an information set $h_j \in H_j$ such that $(t, \hat{t}) \in h_j$ and $\hat{t} < t''$.

To understand this, just think about a few different cases.

1. Assuming first that there is some $t \in P_j$ that precedes $t'$ but no node in $P_j$ that precedes $t''$. Then $j$ would know that it is possible that $t'$ could be reached if $t$ was reached and that $t''$ could not possibly be reached if $j$ has been called to play before reaching the information set. Symmetrically, if player $j$ has not been called to play before reaching the information set the inference would be that $j$ cannot possibly be in node $t'$. 

3
2. Assuming that \( t < t' \) and \( \widehat{t} < t'' \) and that these are in distinct information sets would only be consistent with \((t', t'')\) being in the same information set if the player forgot whether the information set with \( t \) or the one with \( \widehat{t} \) was reached, which is why it makes sense to use the term perfect recall.

### 1.2 Strategies

**Definition 12** A pure strategy for player \( i \) is a mapping \( s_i : H_i \rightarrow \bigcup_{h_i \in H_i} A(h_i) \) such that \( s_i(h_i) \in A(h_i) \) for every \( h_i \in H_i \).

#### 1.2.1 Example 1: Hawk-Dove (Simultaneous)

\[
\begin{array}{cccc}
H & D \\
H & 0,0 & 4,1 \\
D & 1,4 & 3,3 \\
\end{array}
\]

Draw Extensive Form

\( P_1 = t_1 \)
\( P_2 = \{t_2, t_3\} \)
\( H_1 = \{\{t_1\}\} \)
\( H_2 = \{\{t_2, t_3\}\} \)
\( A_1 = S_1 = \{H, D\} \)
\( A_2 = S_2 = \{h, d\} \)

#### 1.2.2 Example 2: Sequential Hawk-Dove

Draw extensive form

\( P_1 = t_1 \)
\( P_2 = \{t_2, t_3\} \)
\( H_1 = \{\{t_1\}\} \)
\( H_2 = \{h_1^1, h_1^2\} = \{\{t_2\}, \{t_3\}\} \)
\( A_1 = S_1 = \{H, D\} \)
\[ A_2 (\{t_2\}) = \{h, d\} \]

\[ A_2 (\{t_3\}) = \{h', d'\} \]

\[ S_2 = \{ s_2 : \{\{t_2\}, \{t_3\}\} \to \{h, d, h', d'\} \text{ such that } s_2 (\{t_2\}) \in \{h, d\} \text{ and } s_2 (\{t_3\}) \in \{h', d'\}\} \]

Clearly, we may represent the strategy set by writing

\[ S_2 = \{ hh', hd', dh', dh' \} \]

where the convention is that the first coordinate in pair \(kl'\) is the action at node \(t_2\) (which follows action \(H\)) and the second is the action at node \(t_3\) which follows action \(D\).

\[ S_2 = \{ s_2 : \{\{t_2\}, \{t_3\}\} \to \{h, d, h', d'\} \text{ such that } s_2 (\{t_2\}) \in \{h, d\} \text{ and } s_2 (\{t_3\}) \in \{h', d'\}\} \]

so we get a Normal form game

\[
\begin{array}{cccc}
hh' & hd' & dh' & dd' \\
H & 0,0 & 0,0 & 4,1 & 4,1 \\
D & 1,4 & 3,3 & 1,4 & 3,3 \\
\end{array}
\]

which has three pure strategy Nash equilibria: \(\{(H, dh'), (H, dd'), (D, hh')\}\).

### 1.3 Moves By Nature

Given a pure strategy profile \(s\) and an extensive game with no moves by nature the extensive form payoffs are translated into normal form payoffs by letting

\[ u_i (s) = u_i (z), \]

where \(z\) is the unique terminal node that is reached when \(s\) is played. If there are moves by nature, let

\[ u_i (s, t) = u_i (z(t)) \]

where \(z(t)\) is the unique terminal node that is reached is \(s\) is played and play begins at node \(t\). The normal form payoffs is thus (note that the notation is somewhat unfortunate in that
$S$ is used for successor nodes as well as for strategies, a problem I am ignoring as we will not have to use the notation for successor nodes much in what follows)

$$u_i(s) = \sum_{t \in S(t_0)} u_i(z(t)) \rho(t).$$

### 1.4 Terminology

- The **outcome path** in a game with no moves by nature is the unique set of nodes are visited when pure strategy profile $s$ is played.

- The **outcome** is the unique terminal node that is reached when $s$ is played.

- In a game with non trivial moves by nature, the **outcome path** is the unique probability distribution over the possible paths implied by $s$ and $\rho$.

- In a game with non trivial moves by nature, the **outcome** is the unique probability distribution over $Z$ implied by $s$ and $\rho$.

- If $s^*$ is an equilibrium we refer to the outcome path as the **equilibrium path**.

- If $s^*$ is an equilibrium we refer to the outcome as the **equilibrium outcome**.

### 1.5 The Reduced Normal Form

#### 1.5.1 The Centipede Game

Consider a short version of Rosenthal’s centipede game where there are 4 non-terminal nodes \{\(t_1, t_2, t_3, t_4\)\} and 5 terminal nodes \{\(z_1, z_2, z_3, z_4, z_5\)\} where;

1. At node $t_1$ player 1 plays an action in $A_1(t_1) = \{s, c\}$ if $s$ is played $z_1$ is reached and the payoffs are $(1, 0)$. If $c$ is played $t_2$ is reached.

2. At node $t_2$ player 2 plays an action in $A_2(t_2) = \{S, C\}$ if $S$ is played $z_2$ is reached and the payoffs are $(0, 10)$. If $C$ is played $t_3$ is reached.
3. At node $t_3$ player 1 plays an action in $A_1(t_3) = \{s', c'\}$ if $s'$ is played $z_3$ is reached and the payoffs are $(100, 1)$. If $c'$ is played $t_4$ is reached.

4. At node $t_4$ player 2 plays an action in $A_2(t_4) = \{S', C'\}$ if $S'$ is played $z_4$ is reached and the payoffs are $(10, 1000)$. If $C'$ is played $z_5$ is reached and the payoffs are $(999, 999)$.

**DRAW EXTENSIVE FORM.**

The available strategies are thus

$$S_1 = \{ss', sc', cs', cc'\}$$
$$S_2 = \{SS', SC', CS', CC'\}$$

and the normal form payoff matrix is

$$\begin{array}{cccc}
SS' & SC' & CS' & CC' \\
ss' & 1,0 & 1,0 & 1,0 & 1,0 \\
sc' & 1,0 & 1,0 & 1,0 & 1,0 \\
\text{cs'} & 0,10 & 0,10 & 100,1 & 100,1 \\
\text{cc'} & 0,10 & 0,10 & 10,1000 & 999,999 \\
\end{array}$$

We observe that:

- Regardless of what player 2 is doing, $ss'$ and $sc'$ result in the same payoffs for both players (because the action $s$ makes all later actions irrelevant).

- Regardless of what player 2 is doing, $SS'$ and $SC'$ result in the same payoffs for both players (because the action $S$ makes all later actions irrelevant).

Hence, we may replace $ss'$ and $sc'$ by $s$ and $SS'$ and $SC'$ by $S$ which gives us the **reduced normal form representation** of the game,

$$\begin{array}{ccc}
S & CS' & CC' \\
\text{s} & 1,0 & 1,0 & 1,0 \\
\text{cs'} & 0,10 & 100,1 & 100,1 \\
\text{cc'} & 0,10 & 10,1000 & 999,999 \\
\end{array}$$
Notice that

\[
\begin{array}{ccc}
S & CS' & CC' \\
\text{s} & 1,0 & 1,0 & 1,0 \\
\text{cs'} & 0,10 & 100,1 & 100,1 \\
\text{cc'} & 0,10 & 10,1000 & 999,999 \\
\end{array}
\]

so that the unique Nash equilibrium in the reduced normal form is \((s, S)\), which gives a payoff of \((1,0)\) despite the fact that \((cc', CC')\) gives a payoff of \((999,999)\).

1.5.2 Twice Repeated Prisoner’s Dilemma

Suppose the rules are as follows. In period 1, the players play

\[
\begin{array}{cc}
D & C \\
d & 0,0 & 2,-1 \\
c & -1,2 & 1,1 \\
\end{array}
\]

After the first period, the players observe the outcome and then play the game a second time.

In repeated games such as this example the information partitions are usually expressed in terms of **histories** instead of the more abstract/general language that we have used so far. That is, instead of being explicit about which particular node belongs to whom we note that all we need to keep track of is the past history of play.

That is, in the first round there is nothing to condition on, which we express as the null history \(h_0\). However, when playing the game in the second period there are 4 different outcomes from the first round of play. These histories are simply all possible outcomes in the first round, so

\[
H_i = \{h_0, dD, dC, cD, cC\}
\]

and the strategy spaces may be expressed as (we will typically not distinguish actions by the different players later on as everything is symmetric)

\[
\begin{align*}
\text{s}_1 & : H_1 \to \{d, c\} \\
\text{s}_2 & : H_2 \to \{D, C\}.
\end{align*}
\]
We note that these strategy spaces are pretty large as each player has \(2^5 = 32\) strategies available. However:

1. Playing \(d\) (\(D\)) in period 1 makes it moot what happens after histories \(cD\) and \(cC\) (\(dC\) and \(cC\))

2. Playing \(c\) (\(C\)) in period 1 makes it moot what happens after histories \(dD\) and \(dC\) (\(dD\) and \(cD\))

Hence, the reduced normal form strategies for player 1 are

\[
d \times \{ f : \{dD, dC\} \to \{d, c\} \} \cup d \times \{ f : \{cD, cC\} \to \{d, c\} \},
\]

which reduces the set of strategies from 32 to 8. If we represent the strategy for player 1 as a triple \(a_1a_2a_3\) where the first coordinate is the action in period 1, the second is the action after \(a_1\) and \(D\) by player 2 and the third is the action after \(a_1\) and \(C\) by player 2 and do the same thing for player 1 we may represent the reduced normal form as:

\[
\begin{array}{cccccccccc}
DDD & DDC & DCD & DCC & CDD & CDC & CCD & CCC \\
\hline
ddd & 0,0 & 0,0 & 2,-1 & 2,-1 & 2,-1 & 2,-1 & 4,-2 & 4,-2 \\
ddc & 0,0 & 0,0 & 2,-1 & 2,-1 & 1,1 & 1,1 & 3,0 & 3,0 \\
dcd & -1,2 & -1,2 & 1,1 & 1,1 & 2,-1 & 2,-1 & 4,-2 & 4,-2 \\
dcc & -1,2 & -1,2 & 1,1 & 1,1 & 1,1 & 1,1 & 3,0 & 3,0 \\
cdd & -1,2 & 1,1 & -1,2 & 1,1 & 1,1 & 3,0 & 1,1 & 0,3 \\
cdc & -1,2 & 1,1 & -1,2 & 1,1 & 0,3 & 2,2 & 0,3 & 2,2 \\
ccd & -2,4 & 0,3 & -2,4 & 0,3 & 1,1 & 3,0 & 1,1 & 3,0 \\
cecc & -2,4 & 0,3 & -2,4 & 0,3 & 0,3 & 2,2 & 0,3 & 2,2 \\
\end{array}
\]

where we see that \((ddd, DDD)\) is the unique Nash equilibrium in the reduced normal form.

## 2 Subgame Perfection

Recall that in the Hawk-Dove game with player 1 moving before player 2 we have that \((D, hh')\) is a Nash equilibrium. Arguably, this is a pretty unintuitive equilibrium as it relies
on that player 1 thinks that player 2 would play $h$ following $H$ despite that this results in a lower payoff than playing $d$ should the node actually be reached. We refer to this as a Nash equilibrium that is supported by a non credible threat.

**Definition 13** A subgame of an extensive form game $K$ is an extensive form game $K'$ where $T' \subset T$ and $\prec', P', A', H', u'$ are defined as in the original game $K$ over the nodes in $T'$, where:

1. $(T', \prec)$ is a valid game tree: there exists a node $t' \in T'$ such that $t' \prec t$ for any $t \in T' \setminus \{t'\}$ and some information set $h_j$ such that $h_j = \{t'\}$.

2. No broken information sets: for every node $t \in T'$, if $t \in h_j$ and $\hat{t} \in h_j$, where $h_j$ is an information set in the original game $K$, then $\hat{t} \in T'$.

**Definition 14** A strategy profile $s^*$ is called a subgame perfect Nash equilibrium if it induces Nash equilibrium play in every subgame $K'$ of $K$.

### 2.1 Subgame Perfection in two Strategically Equivalent Games

#### 2.1.1 Game 1:

- In stage 1, player 1 plays “Out” or “In”.

- If player 1 plays “Out” the payoffs are $(2, 0)$.

- If player 1 plays “In” the simultaneous move game with normal form

$$
\begin{array}{ccc}
c & d \\
 a & 5, 1 & 0, 0 \\
b & 4, 0 & -1, 1 \\
\end{array}
$$

is played.

Draw extensive form.
The associated reduced normal form is

\[
\begin{array}{cc}
c & d \\
\text{Out} & 2, 0 & 2, 0 \\
\text{Ina} & 5, 1 & 0, 0 \\
\text{Inb} & 4, 0 & -1, 1 \\
\end{array}
\]

2.1.2 Game 2:

Consider game with normal form

\[
\begin{array}{cc}
c & d \\
\text{Out} & 2, 0 & 2, 0 \\
\text{a} & 5, 1 & 0, 0 \\
\text{b} & 4, 0 & -1, 1 \\
\end{array}
\]

which has Nash equilibria \((\text{Out}, d), (a, c)\).

2.1.3 Subgame Perfection

- In game 1, \((a, c)\) is the only Nash equilibrium of the subgame. Hence, player one plays In as \(5 > 2\). Thus, \((\text{Ina}, c)\) is the unique subgame perfect equilibrium.

- In game 2, the only subgame is the whole game, so \((\text{Out}, d)\) and \((a, c)\) are both subgame perfect.

- We conclude that subgame perfection can produce different results depending on seemingly irrelevant changes in the extensive form.

2.1.4 Some Remarks About the Logical Foundations of Subgame Perfection.

Again consider the centipede game. We already know that each player stopping at the first time they are called to play is the unique Nash equilibrium in the associated reduced normal form. However, the reduced normal form doesn’t distinguish \(ss'\) from \(sc'\), so let’s solve for the subgame perfect equilibrium.
1. At node 4, player 2 gets 1000 if playing stop and 999 if playing continue. Hence, player 2 stops.

2. At node 3, player 1 gets 100 if playing stop. If playing continue she knows that player 2 will stop, which gives a payoff of 10. Hence, stop is better given that player 2 plays rationally in the final subgame.

3. At node 2, player 2 gets 10 if playing stop and he knows that player 1 will stop in the next round, so stop is optimal given subgame perfect continuation play.

4. Hence, at node 1 player 1 must stop.

We conclude that the unique subgame perfect equilibrium is as follows:

- The subgame perfect equilibrium strategy is to always stop. That is, \((ss', SS')\)

- The subgame perfect equilibrium outcome is that player 1 stops at node 1.

Note however, that the according to this logic player 2 is supposed to play stop at the second node because the player reasons that player 1 will stop at the next node. However, and this is the crux, according to the same logic player 1 should have played stop at the first node, so the second node should never be reached. Hence, if player 2 is called to play there is no consistent theory about player 2.

### 2.2 Backwards Induction

**Definition 15** An extensive form game \(K\) is a game of perfect information if \(P_j = H_j\) for every \(j \in N\).

On the left hand side we have a set of nodes and on the right hand side there is (in general) a set of sets of nodes. Hence, the interpretation is that a game of perfect information is one in which all information sets are singletons.
Theorem 1 (Zermolo-Kuhn) There exists at least one subgame perfect equilibrium (in pure strategies) in any finite game of perfect information.

Proof. Let \{Z, T_1, T_2, ..., \{t_k\}\} be a partition of the nodes so that Z are the terminal nodes, T_1 are immediate precursors of the terminal nodes, T_2 are immediate precursors of a node in T_1. Since the game is finite eventually there is a set T_k where T_k = \{t_k\}, which is the root of the game.

Consider some \(i \in N\) that controls a node \(t \in T_1\). The subgame at such a node is a pure decision theory problem, so she will pick some \(z\) solving

\[
\max_{z \in Z/t \prec z} u(z),
\]

In general, there may be indifferences, but let \(f_1 : T_1 \to Z\) be one solution for each \(t \in T_1\). Consider an equilibrium where in the last stage players play in accordance to the rule \(f_1\). Then, a player at a node in \(T_2\) faces the problem

\[
\max_{t_1 \in T_1/t \prec t_1} u(f_1(t_1))
\]

which again reduces to a pure decision theory problem, which implies that there is at least one maximizer at each node in \(T_2\). Continuing recursively to the root we find that the given that subgame perfect continuation strategies are taken as granted the player at the root will solve a decision problem over the set of terminal nodes which again has at least one solution. ■