A strategy $s_i \in S_i$ is **weakly dominated** in a game $N = (I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I})$ if $\exists s'_i \in S_i$ s.t.

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

and $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$ for some $s_{-i} \in S_{-i}$

$\Rightarrow s'_i$ weakly dominates $s_i$

A strategy is a weakly dominant strategy for $i$ if it weakly dominates every other strategy in $S_i$.

**Second price auction, n bidders**

**Q1** Prove that bidding the valuation is the unique weakly dominant strategy for every player.

**PF To prove, we need to establish both that bidding the valuation is a weakly dominant strategy AND that this weakly dominant strategy is unique.**

Claim: $b_i = V_i$ is a weakly dominant strategy for $i$

$\rightarrow$ To do, NTS that $b_i < V_i$ and $b_i > V_i$ is weakly dominated by $b_i = V_i$.

**Case 1: $b_i > V_i$**

Let $b = \max_{i \neq j} b_i$. There are then 3 cases from the pov of $i$:

1. $b > b_i > V_i$ $\rightarrow$ i would lose, indifferent between bidding $b_i$ and $V_i$.

2. $b_i > b > V_i$ $\rightarrow$ i would win and obtain $V_i - b < 0$ would have done strictly better if bid $b_i = V_i$, lost, and obtained 0.

3. $b_i > V_i > b$

$\rightarrow$ i would win, indifferent between bidding $b_i$ and $V_i$.

So $\exists 1$ realization of $b$ s.t.

$u_i(V_i, b) > u_i(b_i, b)$ and $u_i(V_i, b) \geq u_i(b_i, b)$ for all realizations of $b$.
→ Case 2: \( b_i < v_i \)

Again, there are 3 cases:

1. \( b_i < v_i < b \) → i would lose, indifferent between bidding \( b_i \) and \( v_i \).

2. \( b_i < b < v_i \) → i would lose, obtain 0. If had bid \( b_i = v_i \), would have won and received \( v_i - b > 0 \) and done strictly better.

3. \( b < b_i < v_i \)
   → i would win regardless of bidding \( b_i \) or \( v_i \)
   
   So
   \[ U_i(v_i, b) > U_i(b_i, b) \]
   for 1 realization of \( b \)
   \[ U_i(v_i, b) > U_i(b_i, b) \] ∀ realizations of \( b \)

Thus \( b_i = v_i \) weakly dominates both \( b_i < v_i \) and \( b_i > v_i \)

⇒ claim is proven.  □

Claim: \( b_i = v_i \) is the unique weakly dominant strategy for \( i \)

Suppose that \( \exists b_i \neq v_i \) s.t. \( b_i \) is also a weakly dominant strategy. Then this must mean that \( b_i \) weakly dominates \( v_i \), implying that

\[ U_i(b_i, b_i) > U_i(v_i, b_i) \]

which contradicts \( v_i \) being a weakly dominant strategy.  □
In the second price auction with n bidders, construct a Nash EQ with i winning the object: \( v_i > 0 \) \( \forall i \) arbitrary for each \( i \in \{1, \ldots, n\} \).

In an EQ in which players play weakly dominant strategies, the winner is the player with the highest valuation. What about arbitrary valuations?

**DEF:** \( s^* \) is a NE if \( u_i(s^*) \geq u_i(s_i, s_{-i}) \) \( \forall i \in \mathcal{I} \) \( \forall s_i \in S_i \).

The point is that we can construct a Nash EQ in which i wins the object regardless of what his valuation is.

Let \( v = \max v_i \), \( s = \max s_i \).

Define \( s^* = (s^*_i, s^*_{-i}) \) to be

\[
\begin{align*}
  s^*_i &> v \\
  0 \leq s^*_{-i} &\leq v_i \quad (i.e. \ s^*_i \in [0, v_i])
\end{align*}
\]

Now we try to show that this strategy profile constitutes a Nash EQ.

So now we consider cases. Focus first on player i:

- \( u_i(s^*) = v_i - s \geq 0 \)
- b/c. \( s^*_i > v \geq s^*_{-i} \) \( \begin{cases} \text{so i wins the auction} \\
  \text{gets} \geq 0 \text{ payoff b/c } s^*_{-i} \leq v_i \\
  \Rightarrow s \leq v_i \end{cases} \)

Now take any other \( s_i \neq s^*_i \).

\( \rightarrow \) **Case 1:** \( s_i > s \)

\( u_i(s_i, s^*_{-i}) = v_i - s = u_i(s^*) \)

\( \rightarrow \) **Case 2:** \( s_i < s \)

\( u_i(s_i, s^*_{-i}) = 0 \leq u_i(s^*) \)

Thus, we conclude

\[
\forall s_i \in S_i.
\]

\( u_i(s^*) \geq u_i(s_i, s^*_{-i}) \)
Now focus on players $j \neq i$:

- $u_j(s^*) = 0$ b/c $j$ loses the auction

and also, breaking into cases,

- $s_j > s_i^*$ → $u_j(s_j, s_j^*) = v_j - \max \{s_j^*\} = v_j - s_i^* < 0$

- $s_j < s_i^*$ → $u_j(s_j, s_j^*) = 0$ b/c $j$ loses.

$\Rightarrow u_j(s^*) \geq u_j(s_j, s_j^*) \forall s_j \in S_j$ where $j \neq i$

Conclusion: $s^*$ is a NE where $i$ wins given an arbitrary set of valuations $\{v_i\}_{i \in I}$.
Game with players 1, 2, 3
Alternatives A, B, C
Strategy is a vote \( v \in \{a, b, c\} \)
Alt. K wins if at least two people vote K
If there's a tie, it will be a 3-way tie, each alternative wins with equal probabilities.

\[
\begin{align*}
U_1(A) &= U_2(B) = U_3(C) = 2 \\
U_1(B) &= U_2(C) = U_3(A) = 1 \\
U_1(C) &= U_2(A) = U_3(B) = 0
\end{align*}
\]

Q1) Write down a matrix form.

Note that in the case of a tie, each player gets an expected utility of

\[
\frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = \frac{2}{3} + \frac{1}{3} = 1
\]

The matrix form:

- \( \rightarrow 1 \) chooses rows
- \( \rightarrow 2 \) chooses columns
- \( \rightarrow 3 \) chooses matrix

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2,0,1</td>
<td>2,0,1</td>
<td>2,0,1</td>
</tr>
<tr>
<td>B</td>
<td>2,0,1</td>
<td>1,2,0</td>
<td>1,1,1</td>
</tr>
<tr>
<td>C</td>
<td>2,0,1</td>
<td>1,1,1</td>
<td>0,1,2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2,0,1</td>
<td>1,2,0</td>
<td>1,1,1</td>
</tr>
<tr>
<td>B</td>
<td>1,2,0</td>
<td>1,2,0</td>
<td>1,2,0</td>
</tr>
<tr>
<td>C</td>
<td>1,1,1</td>
<td>1,2,0</td>
<td>0,1,2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2,0,1</td>
<td>1,1,1</td>
<td>0,1,2</td>
</tr>
<tr>
<td>B</td>
<td>1,1,1</td>
<td>1,2,0</td>
<td>0,1,2</td>
</tr>
<tr>
<td>C</td>
<td>0,1,2</td>
<td>0,1,2</td>
<td>0,1,2</td>
</tr>
</tbody>
</table>
(Q4) Pure strategy Nash

The set of Nash EQ

\[ \{ (A, A, A), (A, B, A), (B, B, B), (A, B, C), (A, C, C), (B, B, C), (C, C, C) \} \]

(Q2) Weakly dominant strategies?
Strictly dominant strategies?

Claim:  
1. \( S_1 = A \) is unique weakly dominant strategy for player 1
2. \( S_2 = B \)
3. \( S_3 = B \)

PF

We prove 1; 2 and 3 follow similarly. Basically what we need to do is show that given any realization of \( S_1 = (S_2, S_3) \), 1 can always do weakly better by voting A. Thus:

- \( U_1(A, S_2, S_3) > U_1(B, S_2, S_3) \)

  when either \( S_2 = A \) or \( S_3 = A \) but not both.

  (so \( B \) cannot be weakly dominant also)

- \( U_1(A, S_2, S_3) > U_1(C, S_2, S_3) \)

  when either \( S_2 = A \) or \( S_3 = A \) but not both

  (so \( C \) cannot be weakly dominant also)

- \( S_2 = S_3 = A \),
  \[ U_1(A, A, A) = U_1(B, A, A) = U_1(C, A, A) \]  

- \( S_2 = S_3 = B \)
  \[ U_1(A, B, B) = U_1(B, B, B) = U_1(C, B, B) \]
\[ S_2 = S_3 = C \]

\[ U_A(A, C, C) = U_1(B, C, C) = U_2(C, C, C) \quad (0) \]

\[ U_1(A, S_2, S_3) = U_1(B, S_2, S_3) = U_2(C, S_2, S_3) \]

\[ \Rightarrow S_2 = S_3 \text{ basicall when } S_2 = S_3, \]

\[ U_1(A, S_2, S_3) = U_1(B, S_2, S_3) = U_2(C, S_2, S_3) \]

- \[ U_1(A, B, C) = U_1(B, B, C) > U_1(C, B, C) \]
- \[ U_1(A, C, B) = U_1(B, C, B) > U_1(C, C, B) \]

**Conclusion:** For any realization of votes \( S_{-1} \),

\[ U_1(A, S_{-1}) \geq U_1(S_1, S_{-1}). \]

And for some realization of votes \( S_{-1} \)

\[ U_1(A, S_{-1}) > U_1(S_1, S_{-1}) \quad \forall S_1 \in \{ A, B, C \} \]

\[ \Rightarrow A \text{ is weakly dominant for player } 1. \]

**Claim:** There are no strictly dominant strategies

\[ \text{PF} \]

To prove this, focus on \( P1 \) (other players' cases follow similarly).

We know that \( B + C \) are weakly dominated by \( A \)

\[ \Rightarrow B \text{ and } C \text{ cannot be strictly dominant. Also,} \]

\[ U_1(A, A, A) = U_2(B, A, A) = U_3(C, A, A) \]

\[ \Rightarrow A \text{ is not strictly dominant. } (\exists S_{-1} \in S_{-1} \text{ s.t.} \]

\[ U_1(A, S_{-1}) \not> U_1(B, S_{-1}) = u_1(C, S_{-1}) \]

\[ \text{ widetilde{}} \]
Again, just focus on P1: (P2, P3 follow similarly)

A is weakly dominated for P1

⇒ B, C are weakly dominated by A for P1.

Claim: There are no strictly dominated strategies for P1

PF: (contradiction)

• Suppose A strictly dominates B. But
  \[ U_1(A, A, A) = U_1(B, A, A) \] (also shows B does not strictly dominate A)

• Suppose A strictly dominates C. But
  \[ U_1(A, A, A) = U_1(C, A, A) \] (also shows that C does not strictly dominate A)

• Suppose B strictly dominates C. But
  \[ U_1(B, A, B) = U_1(C, A, B) \] (also shows that C does not strictly dominate B)

\[ \square \]

Note:

• It's always true that if a player has a strictly dominant strategy ⇒ all of player's other strategies are dominated.

• But there may be cases where a player does not have a dominant strategy and yet has dominated strategies.

\[ \begin{array}{c|cc}
  & A & B \\
  \hline
  A & 5,2 & 4,2 \\
  B & 3,1 & 3,2 \\
  C & 2,1 & 4,1 \\
  D & 4,3 & 5,4 \\
\end{array} \]

No strictly dominant strategies for 1.
But A strictly dominates B for P1

For example,

\[ U_1(A, A) > U_1(D, A) \]
but \[ U_1(A, B) < U_1(D, B) \]

\[ \square \]
3) Let Alice - row player
    Bob - column player
    Curt - matrix player

Let \( W = \) work
\( S = \) shirk

Q1)

\[
\begin{array}{ccc}
 & W & S \\
W & 1, 1, 1 & \frac{1}{2}, \frac{3}{2}, 1 \\
S & \frac{3}{2}, 1, -\frac{1}{2} & 0, \frac{3}{2}, -\frac{1}{2}
\end{array}
\]

Q2) Find any dominant or dominated strategies.

PF)
let's go one by one, by player.

Alice
Alice's payoffs only depend on her strategy and Bob's.
Thus:
\[
\begin{align*}
U_A (S, W, -) &= \frac{3}{2} > U_A (W, W, -) = 1 \\
U_A (S, S, -) &= 0 > U_A (W, S, -1) = -\frac{1}{2}
\end{align*}
\]
So: \( S \) strictly dominates \( W \) for Alice.
Bob: B's payoffs depend on only Bob and Curt's strategies. Thus:

\[ U_B (-, S, W) = \frac{3}{2} > U_B (-, W, W) = 1 \]
\[ u_B (-, S, S) = 0 > u_B (-, W, S) = -\frac{1}{2} \]

So S strictly dominates W for Bob. \( \Box \)

Curt: C's payoffs only depend on Alice and Curt.

\[ U_C (S, -, S) = 0 > U_C (S, -, W) = -\frac{1}{2} \]
\[ U_C (W, -, S) = \frac{3}{2} > U_C (W, -, W) = 1 \]

So S strictly dominates W for Bob. \( \Box \)
4) Cournot Duopoly

\[ p(q) = 2 - q \] (inverse demand)
\[ c + c = 1 \]

Q1) Normal form:
\[ I = \{ i, j \} \]
\[ q_i, q_j \in [0, \infty) \forall i, j \in I \]
\[ \Pi_i(q_i, q_j) = q_i (p - c) \]
\[ = q_i (2 - q_i - q_j - c) \]

Q2) Solve by iterated elimination of strictly dominated strategies, illustrate in graph.

Note that the unique maximizer to \( \mathcal{O} \) is
\[
\frac{\partial \Pi_i(q_i, q_j)}{\partial q_i} = 0 \Rightarrow (2 - q_i - q_j - c) + q_i (-1) = 0
\]
\[ \Rightarrow 2 - q_j - c - 2q_i = 0 \]
\[ \Rightarrow \frac{2 - q_j - c}{2} = q_i \]
So monopoly output is \[ \frac{2 - c}{2} = q_m \]

\[ q_m = \frac{1}{2} \]
If we plot out the best responses, we have (for a general $c$)

Proced by steps:

1. $q_1, q_2 \in \left[0, \frac{1}{2}\right]$ b/c no firm would produce above monopoly output. Each firm knows this, so the best they could do if the other firm produces at monopoly output is

   Thus: $q_1, q_2 \geq \frac{1 - \frac{1}{2}}{2} = \frac{1}{4}$

2. So still have $q_1, q_2 \in \left[\frac{1}{4}, \frac{1}{2}\right]$. Note that playing $q_1 = \frac{1}{4}$ is strictly dominated by playing $q_1 = \frac{1}{4} q_2$, so $q_2 \leq \frac{1 - \frac{1}{4}}{2} = \frac{3}{8}$

   So: $q_1, q_2 \in \left[\frac{1}{4}, \frac{3}{8}\right]$

3. Again, we have

   $q_1, q_2 \geq \frac{1 - \frac{3}{8}}{2} = \frac{5}{16}$

   So: $q_1, q_2 \in \left[\frac{5}{16}, \frac{3}{8}\right]$
Continuing, we have:

\[ a_n = \frac{1 - b_{n-1}}{2}, \quad b_n = \frac{1 - a_{n-1}}{2} \quad \text{where} \quad a_0 = b_0 = 0 \]

In the limit, we then have:

\[ a_\infty = 1 - \frac{1 - a_\infty}{2} \Rightarrow \begin{cases} a_\infty = \frac{1}{3} \\ b_\infty = \frac{1}{3} \end{cases} \]
5 Setup

\( n \) agents
endowment \( \to 1 \) unit of private good.
Out of this endowment, agents choose to contribute \( g_i \) to the provision of the public good, where \( g_i > 0 \)
Technology is \( s.t \ G = \sum_{i=1}^{n} g_i \)
utility is \( u_i(x_i; G) \)

Firstly, note that
\( x_i = 1 - g_i \)

So the utility function may be written as
\( u_i(1-g_i, g_i + G_{-i}) \) where \( G_{-i} = \sum_{j \neq i} g_j \)

Q1 Normal Form:
\[ I = \{1, \ldots, n\} \]
\[ G_i = (0, 1] \quad \forall i \Rightarrow G = \sum_{i=1}^{n} G_i \]
The payoff function for \( i \) is
\[ p_i(\cdot, g_{-i}) = u_i(1-g_i, g_i + G_{-i}) \]
a function \( p_i: G \to \mathbb{R} \)
where \( p = (p_1(\cdot), \ldots, p_n(\cdot)) \)

Q2 First of all,
* A Nash Equilibrium is a strategy profile \( g^* = (g_1^*, \ldots, g_n^*) \) s.t
\[ g_i^* \in \text{arg max } u_i(1-g_i, g_i + G_{-i}) \quad \forall i \in I \]

Given
\[ u_i(x_i; G) = x_i - (1-G)^2 \]
we may convert the utility function into a payoff function by substituting in appropriate terms:
\[ p_i(g_i, g_{-i}) = (1-g_i) - \left(1 - \sum_{i=1}^{n} g_i\right)^2 \]
\[ = (1-g_i) - (1 - g_i - G_{-i})^2 \]
The payoff function is concave in $g_i \Rightarrow$ take derivatives, find FOC.

\[
\frac{\partial p_i(g_i,g_{-i})}{\partial g_i} = 0 \Rightarrow -1 - 2(1-g_i-G_{-i}) - 1 = 0
\]

\[
\Rightarrow 2(1-g_i-G_{-i}) = 1
\]

\[
\Rightarrow (1-g_i-G_{-i}) = \frac{1}{2}
\]

\[
\Rightarrow \frac{1}{2} - G_{-i} = g_i \tag{\star}
\]

Thus a Nash EQ is a strategy profile $g^* = (g_1^*, \ldots, g_n^*)$ s.t

\[\tag{\star} \text{ holds } \forall i \text{ and } g_i \in (0,1] \quad \forall i.\]

\[\Phi\]

Similarly, rewriting we can get

\[
u_i(x_i, G) = \alpha_i \ln(x_i) + (1-\alpha_i) \ln(G)
\]

\[
p_i(g_i,g_{-i}) = \alpha_i \ln(1-g_i) + (1-\alpha_i) \ln(g_i + G_{-i})
\]

\[
\frac{\partial p_i(g_i,g_{-i})}{\partial g_i} = 0 \Rightarrow -\frac{\alpha_i}{1-g_i} + \frac{1-\alpha_i}{g_i + G_{-i}} = 0
\]

\[
\Rightarrow \frac{(1-\alpha_i)}{g_i + G_{-i}} = \frac{\alpha_i}{1-g_i} \tag{\star}
\]

Thus a Nash EQ is a strategy profile $g^* = (g_1^*, \ldots, g_n^*)$ s.t

\[\tag{\star} \text{ holds } \forall i \text{ and } g_i \in (0,1] \quad \forall i.\]

\[\Omega\]