Find all Nash EQ

**Steps:**

1. Check for pure-strategy NE: None.
2. Check for mixed strategy NE.

There are two cases for each player. Each player could either mix between two or three strategies.

**Case 1:** Players mix between two strategies

Suppose Bart places positive probability on only R and S (i.e., \( p_1 + p_2 = 1 \)). Then Lisa's expected payoffs given this probability distribution used for Bart is

- \( U_L(R) = p_2 = 1 - p_1 \)
- \( U_L(S) = -p_1 \)
- \( U_L(P) = p_1 + p_2 = 2p_1 - 1 \)

From these expected utilities, note that Lisa would never play S (since she can do better by just playing R. She will play a mixed strategy that places positive probability on playing R and P if

\[ 1 - p_1 = 2p_2 - 1 \Rightarrow p_2 = \frac{2}{3}, p_1 = \frac{1}{3} \]

So then Bart is willing to mix only between R and S if

\[ U_B(R; q_1, 0, 1-q_2) = U_B(S; 0, 1, 0) \]

\[ \Rightarrow \alpha q_1 + (\alpha - 1)(1-q_2) = -q_1 + (1-q_2) \]

\[ \Rightarrow \alpha q_1 + (\alpha - 1) - (\alpha - 1)q_2 = 1 - 2q_2 \]

\[ \Rightarrow \alpha q_1 + (\alpha - 1) - \alpha q_1 + q_1 = 1 - 2q_2 \]

\[ \Rightarrow 3q_1 = 2 - \alpha \Rightarrow q_1 = \frac{2-\alpha}{3} \]
What would Bart get if he had played P?

\[ U_B(P; q_1, 0, 1-q_1) = q_1 \]

Since Bart does not play P with positive probability, it's assumed that

\[ U_B(R) = U_B(S) > U_B(P) \]

\[ \alpha - (1-q_1) > q_1 \]

\[ \Rightarrow \alpha > 1, \text{ which cannot happen since } \alpha \in [0, 1]. \]

So we have that

L \Rightarrow Bart does not want to mix between only R and S when Lisa doesn't put any weight on S.

L \Rightarrow But Lisa only doesn't put weight on S if Bart only mixes over R and S

\[ \Rightarrow \text{This cannot be an EQ.} \]

The same reasoning holds for other cases where players use positive probability to mix between only 2 strategies.

**Case 2:** Players mix (with positive probability) amongst all strategies

Supposing Lisa plays \( \sigma_L = (q_1, q_2, 1-q_1-q_2) \) and Bart plays \( \sigma_B = (p_1, p_2, 1-p_1-p_2) \), we have that

\[ U_L(R; \sigma_B) = 0 \cdot p_1 + 1 \cdot p_2 + (-1)(1-p_1-p_2) = 2p_2 + p_1 - 1 \]

\[ U_L(S; \sigma_B) = (-1) \cdot p_1 + 0 \cdot p_2 + 1 \cdot (1-p_1-p_2) = 1-2p_1 - p_2 \]

\[ U_L(S; \sigma_B) = 1 \cdot p_1 + (-1)p_2 + 0(1-p_1-p_2) = p_1 - p_2 \]

Indifference condition

\[ U_L(R) = U_L(S) = U_L(S) \]

\[ \Rightarrow 2p_2 + p_1 - 1 = 1-2p_1 - p_2 = p_1 - p_2 \]

\[ \Rightarrow p_1 = p_2 = \frac{1}{3} \]

So: \( \sigma_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \)
Similarly for Bart,

\[ U_R(R; s_L) = \alpha q_1 + (\alpha + 1) q_2 + (\alpha - 1) (1 - q_1 - q_2) = \alpha - 1 + q_4 + 2 \]

\[ U_R(S; s_L) = -q_2 + (1 - q_1 - q_2) \]

\[ U_R(P; s_L) = q_1 - q_2 \]

Thus the indifference condition is

\[ \alpha - 1 + q_1 + 2q_2 = 1 - 2q_1 - q_2 = q_1 - q_2 \]

\[ \Rightarrow q_4 = \frac{1}{3}, q_2 = \frac{1 - \alpha}{3}, 1 - q_1 - q_2 = \frac{1 + \alpha}{3} \]

**CONCLUSION:**

The set of NE is a singleton:

\[ \text{NE} = \left\{ \left( s_B, s_L \right) : s_B = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), s_L = \left( \frac{1}{3}, \frac{1 - \alpha}{3}, \frac{1 + \alpha}{3} \right) \right\} \]
2. Infinite repetition of the stage game

\[
\begin{array}{c|cc}
\text{(p) } A_1 & A_2 (q) & B_2 (1-q) \\
\hline
8, 8 & 1, 2 \\
2, 1 & 0, 0 \\
\end{array}
\]

1) Find the set of NE EQ:

Pure strategy NE = \{ (A_1, A_2) \}

Are there any mixed strategy NE? Suppose 1 and 2 mix with the above probabilities. Then it must be that for player 1:

\[
8q + (1-q) = 2q
\]
\[
\Rightarrow 2q + 1 - q = 2q
\]
\[
\Rightarrow 7q + 1 = 2q
\]
\[
\Rightarrow q < 0 \text{ (can never happen)}
\]

By symmetry, this cannot work for player 2 either.

\[\Rightarrow\] No mixed strategy NE.

In particular, a general result is that

No strictly dominated strategy will be played with positive probability in a mixed strategy EQ

(\(A_1\) strictly dominates \(B_1\), \(A_2\) strictly dominates \(B_2\)).
2) For a large enough $\varepsilon$, construct a subgame perfect EQ where $(A_1, B_2)$ is played in each round.

Construct the strategy given by this automaton:

$$
\begin{array}{c}
\text{Start} \\
(A_1, A_2) \\
(B_1, A_2) \\
(B_1, B_2) \\
(A_1, B_2)
\end{array}
\quad
\begin{array}{c}
\text{A_2} \\
A_1 \quad 8,8 \\
1,2
\end{array}
\quad
\begin{array}{c}
\text{B_2} \\
B_1 \quad 2,1 \\
0,0
\end{array}
$$

Basically, we will play $(A_1, B_2)$ forever on the EQ path of play, and if $2$ deviates by playing $A_2$ instead of $B_2$, we will move to $(B_1, A_2)$ and stay there unless $1$ deviates and plays $A_1$ instead of $B_1$, in which case we move to $(A_1, B_2)$.

So EQ path of play is $(A_1, B_2), (A_1, B_2), (A_1, B_2), \ldots$.

Check for deviations (one-shot) on path of play:

**P4**: After $h_t = (A_1, B_2), (A_1, B_2), \ldots$

- 1 deviates to induce $(B_1, B_2)$ to receive 0 in $t$ and 1 forever after. If he doesn't deviate he gets 1 forever.

$$
\frac{1}{1-\varepsilon} > 0 + \frac{3,1}{1-\varepsilon}
\Rightarrow 1 > 2 \text{ (always true)}
$$

**P2**: 1 deviates after $h_t = (A_1, B_2), (A_2, B_2), \ldots$

- by playing $A_2$ and so gets 8 in $t$ and 1 forever.

If he had stuck with the prescribed strategy he'd get 2 forever.

$$
\frac{2}{1-\varepsilon} > 8 + \frac{6}{1-\varepsilon}
\Rightarrow 2-\varepsilon > 8-8\varepsilon
\Rightarrow \frac{7\varepsilon}{8} > \frac{8}{7+4}
$$
So for deviations on the path of play, we see that when $\theta \geq 6/7$, we have that the strategy is Nash.

Now, we must check for histories $h_t$ following

- $(A_1, A_2)$
- $(B_1, B_2)$
- $(B_1, A_2)$

in order to check for SPE.

**Histories following $(B_1, B_2)$**

If $(B_1, B_2)$ occurred, it is either because 1 deviated from $(A_1, B_2)$ or that 2 deviated from $(B_1, A_2)$. In the following, we check only for 1 (2 follows similarly).

- If 1 follows prescribed strategy, she'll get $1/\theta$ (play $(A_1, B_2)$ forever).
  - **If 1 deviates again she gets $0 + \frac{\theta}{1-\theta}$**
    - Clearly, $\frac{1}{1-\theta} > 0 + \frac{\theta}{1-\theta} \Rightarrow 1 > \theta \Rightarrow$ no profitable deviation for 1.

- If 2 plays prescribed strategy, 2 gets $\frac{2}{1-\theta}$ (play $(A_1, B_2)$ forever).
- Else, if 2 deviates, 2 plays $A_2$, which induces $(B_1, A_2)$ and stays there forever. So clearly
  \[ \frac{2}{1-\theta} \geq \frac{2}{1-\theta} \Rightarrow \text{no profitable deviation for } 2. \]

**Histories following $(B_1, A_2)$**

- If 1 deviates, 1 gets $8 + \frac{3}{1-\theta}$
  - If 1 follows strategy, 1 gets $\frac{2}{1-\theta}$
    - \[ \frac{2}{1-\theta} > 8 + \frac{3}{1-\theta} \Rightarrow 2 - 8 \geq 8 - 8 \theta \Rightarrow \theta \geq 6/7 \]

- If 2 deviates, 2 gets $0 + \frac{\theta}{1-\theta}$
  - If 2 follows strategy, 2 gets $\frac{1}{1-\theta}$
    - \[ \frac{1}{1-\theta} > 0 + \frac{\theta}{1-\theta} \Rightarrow 1 > \theta \quad \text{(always true)} \]
Histories Following \((A_1, A_2)\)

There are two sorts of histories following \((A_1, A_2)\)  

**Case 1:** 1 deviated, inducing \((A_1, A_2)\) [now move to \((A_1, B_2)\)]

- If 1 deviates again, he induces \((B_1, B_2)\) for 1 period, then \((A_1, B_2)\) forever
- If 1 follows prescribed strategy, he gets \(\frac{1}{1-\delta}\)
  \[\Rightarrow \frac{1}{1-\delta} > 0 + \frac{\delta}{1-\delta} \Rightarrow 1 > \delta \quad \text{(always true)}\]

- If 2 deviates, he obtains \(8 + \frac{\delta}{1-\delta}\)
  - If 2 follows prescribed strategy, he gets \(\frac{2}{1-\delta}\)
    \[\Rightarrow \frac{2}{1-\delta} > 8 + \frac{\delta}{1-\delta}\]
    \[\Rightarrow 2 - 8 > \frac{\delta}{1-\delta}\]
    \[\Rightarrow 2 > \frac{\delta}{1-\delta}\]
    \[\Rightarrow 2 > \frac{6}{7}\]

**Case 2:** 2 deviated, inducing \((A_1, A_2)\) [now move to \((B_1, A_2)\)]

- If 2 deviates again, he gets \(0 + \frac{\delta}{1-\delta}\)
  - If he follows prescribed strategy, he gets \(\frac{2}{1-\delta}\)
    \[\Rightarrow 1 > \delta \quad \text{always true,} \]

- If 1 deviates, 1 gets \(8 + \frac{\delta}{1-\delta}\)
  - If 1 follows proposed strategy, gets \(\frac{2}{1-\delta}\)
    \[\Rightarrow \frac{2}{1-\delta} > 8 + \frac{\delta}{1-\delta} \Rightarrow 2 > \frac{6}{7}\]
Conclusion

With $\alpha \geq \frac{6}{7}$, the proposed strategy profile induces $(A_1, B_2)$ being played in each period and is SPNE.
Consider the Hawk–Dove game:

<table>
<thead>
<tr>
<th></th>
<th>H(\rho)</th>
<th>D(1-\rho)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H(\rho)</td>
<td>\frac{1}{2}(v-c), \frac{1}{2}(v-c)</td>
<td>V, 0</td>
</tr>
<tr>
<td>D(1-\rho)</td>
<td>0, V</td>
<td>\frac{V}{2}, \frac{V}{2}</td>
</tr>
</tbody>
</table>

1) Find the best-correspondence \( (V, C) \) pairs (\( V, C \))

The game is symmetric.

We have the following expected utilities:

\[ U_i(H) = \rho \cdot \frac{1}{2}(v-c) + (1-\rho) \cdot V \]

\[ U_i(D) = \rho \cdot 0 + (1-\rho) \cdot \frac{V}{2} = (1-\rho) \frac{V}{2} \]

Assuming \( i \) is using a mixed strategy (and so chooses \( \sigma_i = (\sigma_i(H), \sigma_i(D)) \)) we have

\[ BR_i(\sigma_i = (\rho, 1-\rho)) = \begin{cases} 
\{1\} & U_i(H) = \rho \cdot \frac{1}{2}(v-c) + (1-\rho) \cdot V > U_i(D) = (1-\rho) \frac{V}{2} \\
[0, 1] & \rho \cdot \frac{1}{2}(v-c) + (1-\rho) \cdot V = (1-\rho) \frac{V}{2} \\
\{0\} & \rho \cdot \frac{1}{2}(v-c) + (1-\rho) \cdot V < (1-\rho) \frac{V}{2} 
\end{cases} \]

Simplifying the conditions on the RHS, we have

\[ \rho \cdot \frac{1}{2}(v-c) + (1-\rho) \cdot V = (1-\rho) \frac{V}{2} \]

\[ \Rightarrow \quad \frac{V}{c} = \rho \]
Thus the best-response correspondence is:

\[
BR_i(g_j) = \begin{cases} 
\{1\} & \frac{V}{c} > p \\
[0,1] & \frac{V}{c} = p \\
\{0\} & \frac{V}{c} < p 
\end{cases}
\]

Symmetric for i,j
2) Given that \( V < C \), find all NE.

\[
V < C \Rightarrow \frac{V}{C} < 1
\]

Can draw the BR correspondence as drawn below.

There are 3 NE strategy profiles (i.e., the Best Response Correspondences intersect twice)

\[
NE = \left\{ (H, D), (D, H), \left( \frac{V}{C}, \frac{1-V}{C} ; \frac{V}{C}, 1-\frac{V}{C} \right) \right\}
\]

3) The game is symmetric \( \rightarrow \) payoffs don’t depend on whether or not you are a column or row player

\( \rightarrow \) Nash will be played (assuming players are rational).
4) The extensive form of the game looks like the following (in $T=3$)

Each player has 3 information sets, 2 actions at each
$\Rightarrow 2^3 = 8$ strategies

Denote a strategy for player $i$ as a 3-tuple $(S_1, S_2, S_3)$

what player plays in information set 1 (i.e. in 1st period)
We need to find the reduced normal form of the game. To do that, we need to eliminate strategies that give the same payoff.

<table>
<thead>
<tr>
<th></th>
<th>CCC</th>
<th>CCD</th>
<th>CDC</th>
<th>DCC</th>
<th>DDC</th>
<th>DDD</th>
<th>CDD</th>
<th>DCD</th>
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<tbody>
<tr>
<td>CCC</td>
<td>6,6</td>
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<td>-1,3</td>
<td>-1,3</td>
<td>1,5</td>
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<td>CDC</td>
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<td>5,1</td>
<td>2,2</td>
<td>-1,3</td>
<td>-1,3</td>
<td>-1,3</td>
<td>2,2</td>
<td>-1,3</td>
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<tr>
<td>DCC</td>
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<td>0,0</td>
<td>0,0</td>
<td>3,1</td>
<td>0,0</td>
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<tr>
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<td>0,0</td>
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<td>-1,3</td>
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<td>2,2</td>
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<td>3,1</td>
<td>3,1</td>
<td>0,0</td>
<td>0,0</td>
<td>0,0</td>
<td>3,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

So there are 4 groups of strategies:

1. DCC
2. CDC
3. CCC (no deviations)
4. CCD (1st deviation is in last period)

(1st deviation is in time 1)

(2nd deviation is in time 2)
So reduced normal form:

<table>
<thead>
<tr>
<th></th>
<th>DXX</th>
<th>CDX</th>
<th>CCD</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>DXX</td>
<td>(0, 0)</td>
<td>3, -1</td>
<td>3, -1</td>
<td>3, -1</td>
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<tr>
<td>CDX</td>
<td>-1, 3</td>
<td>2, 2</td>
<td>5, 1</td>
<td>5, 1</td>
</tr>
<tr>
<td>CCD</td>
<td>-1, 3</td>
<td>1, 5</td>
<td>4, 4</td>
<td>7, 3</td>
</tr>
<tr>
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<td>-1, 3</td>
<td>1, 5</td>
<td>3, 7</td>
<td>6, 6</td>
</tr>
</tbody>
</table>
2) Find the set of all Nash EQ.

The set of all Nash EQ is extremely large. In particular,

\[ \text{NE} = \left\{ (DXX, DXX) : X \in \{C, D\} \right\} \]

3) For an arbitrary \( T < \infty \), find all NE of the game.

Generalizing, we have:

\[ \text{NE} = \left\{ \begin{array}{l}
(DXXX...X, DXXX...X) = X \in \{C, D\} \\
\text{where} \; X = \frac{2}{1-\delta} \\frac{2}{1-\delta}
\end{array} \right\} \]

4) Infinite horizon: Construct two EQ with diff payoffs where \( \frac{1}{3} < \delta < 1 \).

\[ \boxed{\text{EQ 1: Nash Reversion, yielding} \; \frac{2}{1-\delta} \; \text{to each player}} \]

\[ \begin{align*}
S_i(h_0) &= C \\
S_i(h_t) &= \begin{cases} C & \text{if } h_t = ((C, C))_t \\
D & \text{otherwise}
\end{cases}
\end{align*} \]

This strategy is Nash. CHECK for deviations on EQ path of play:

Player i gets \( \frac{2}{1-\delta} \) if follows proposed strategy.

If deviates, i gets \( 3 + \frac{0.6}{1-\delta} \)

This is Nash if \( \frac{2}{1-\delta} \geq 3 \Rightarrow 2 \geq 3 - 3\delta \Rightarrow 3\delta > 1 \Rightarrow \delta > \frac{1}{3} \)

Thus EQ 1 is valid. \( \square \)
EQ 2): The easy one of playing (0,0) in first period, anything else afterwards.
Yields payoff of 0 to both periods.
This is Nash. Why?
Check: deviations off EQ path.
If any player i deviates, i gets -1 < 0 and the game ends. There are no profitable deviations.